A HIERARCHY OF RELAXATION MODELS FOR TWO-PHASE FLOW

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Abstract. A hierarchy of relaxation two-phase flow models is considered, formulated as hyperbolic relaxation systems with source terms. The relaxation terms cause volume, heat and mass transfer due to differences in pressure, temperature and chemical potential, respectively, between the two phases.

The subcharacteristic condition is a concept closely related to the stability of such relaxation systems. It states that the wave speeds of an equilibrium system never can exceed the speeds of the corresponding relaxation system. The work of Flåtten and Lund [*Math. Mod. and Meth. in Appl. Sciences*, 21(12), 2011, pp. 2379-2407] is extended, with analytical expressions for the wave velocities in each model in the mentioned hierarchy. The subcharacteristic condition is explicitly shown to be satisfied using sums of squares, subject only to physically fundamental assumptions.

Key words. Subcharacteristic condition; relaxation; two-phase flow

AMS subject classifications. 76T10, 35L60

1. Introduction. Two-phase flow is found in many industrial applications, such as nuclear reactors [6], heat exchangers, petroleum production [4] and carbon dioxide capture, transport and storage (CCS) [5]. Modelling such flow for use in simulations is a challenging task due to the complex nature of the interactions between the two phases, such as the movement and shape of the interface, and heat and mass transfer across it. In cases where the precise shape of the interface is of less importance or too computationally expensive to calculate, one may apply *averaging* (see e.g. Ishii and Hibiki [13]) of the quantities of the two-phase fluid over a certain area or volume. These averaged models can often be formulated as hyperbolic relaxation systems with source terms accounting for the phase interactions, in the form

$$\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{A}(\boldsymbol{U})\frac{\partial \boldsymbol{U}}{\partial x} + \frac{1}{\varepsilon}\boldsymbol{R}(\boldsymbol{U}) = 0, \tag{1.1}$$

where $\boldsymbol{U} \in \mathbb{R}^n$ is the vector of unknowns, and ε is a characteristic time for the relaxation process described by $\boldsymbol{R}(\boldsymbol{U})$. The hyperbolicity requires that the $n \times n$ matrix $\boldsymbol{A}(\boldsymbol{U})$ is diagonalizable with real eigenvalues. Such relaxation systems have been analysed by Chen et al. [7], Liu [18] and Yong [29]. For a further review of the literature on such systems, see e.g. Natalini [21].

We now assume that there exists a constant $k \times n$ matrix $\boldsymbol{P}$ associated with $\boldsymbol{R}$ which has the property that

$$\boldsymbol{P}\boldsymbol{R}(\boldsymbol{U}) = 0. \tag{1.2}$$

By multiplying Eq. (1.1) with $\boldsymbol{P}$ on the left, we get an equation system for the reduced variables $\boldsymbol{u} = \boldsymbol{P}\boldsymbol{U}$,

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{P}\boldsymbol{A}(\boldsymbol{U})\frac{\partial \boldsymbol{U}}{\partial x} = 0. \tag{1.3}$$

We now make the assumption that $\boldsymbol{u}$ determines an equilibrium value $\boldsymbol{U} = \mathcal{E}(\boldsymbol{u})$ such that $\boldsymbol{R}(\mathcal{E}(\boldsymbol{u})) = 0$ and

$$\boldsymbol{P}\mathcal{E}(\boldsymbol{u}) = \boldsymbol{u}. \tag{1.4}$$

We finally assume $\boldsymbol{u}$ to be sufficiently smooth, so that we may formulate a quasi-linear *equilibrium* system as

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{B}(\boldsymbol{u})\frac{\partial \boldsymbol{u}}{\partial x} = 0, \tag{1.5}$$

$$\boldsymbol{U} = \mathcal{E}(\boldsymbol{u}), \tag{1.6}$$

where $\boldsymbol{B}(\boldsymbol{u}) = \boldsymbol{P}\boldsymbol{A}(\mathcal{E}(\boldsymbol{u}))\partial_{\boldsymbol{u}}\mathcal{E}(\boldsymbol{u})$. As the relaxation time ε of the relaxation system (1.1) goes to zero, we expect the solutions to approach the solutions of the equilibrium system (1.5).

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1.1. The subcharacteristic condition. The subcharacteristic condition is a concept which has proven to be closely related to the stability of relaxation systems. This was first mentioned by Whitham [28] for the linear case, and later developed for 2×2 non-linear systems by Liu [18]. A similar condition was also discussed by Leray [17]. For more general systems, Yong [29] introduced a *relaxation criterion*, which imposes a certain stability requirement on the (linearized) relaxation system and requires that the relaxation term $\boldsymbol{R}(\boldsymbol{U})$ be nonoscillatory, and showed that for $k = n-1$ this criterion leads to a) convergence of the solution in the limit $\varepsilon \to 0$, and b) the subcharacteristic condition being fulfilled.

The subcharacteristic condition has also proven to be an important trait of many physically revelant models. For this reason, the literature on relaxation systems puts a strong emphasis on this condition, see e.g. Baudin [2, 3] and Flåtten [11].

In the context of our relaxation system (1.1) and the corresponding equilibrium system (1.5), the subcharacteristic condition can be defined as follows.

DEFINITION 1. *Let the eigenvalues of the matrix* $\boldsymbol{A}(\boldsymbol{U})$ *of the relaxation system* (1.1) *be given by*

$$\Lambda_1 \leq \ldots \leq \Lambda_i \leq \Lambda_{i+1} \leq \ldots \Lambda_n. \tag{1.7}$$

Similarly, let the eigenvalues of the matrix $\boldsymbol{B}(\boldsymbol{u})$ *of the equilibrium system* (1.5) *be given by*

$$\lambda_1 \leq \ldots \leq \lambda_i \leq \lambda_{i+1} \leq \ldots \lambda_k. \tag{1.8}$$

Also let the equilibrium system's eigenvalues λ_i *be* interlaced *with the relaxation system's eigenvalues, in the sense that* $\lambda_i \in [\Lambda_i, \Lambda_{i+n-k}]$. *Here, the relaxation eigenvalues* Λ_i *are evaluated in an equilibrium state such that*

$$\Lambda_i = \Lambda_i(\mathcal{E}(\boldsymbol{u})), \qquad \lambda_i = \lambda_i(\boldsymbol{u}) \tag{1.9}$$

Then the equilibrium system (1.5) *is said to satisfy the* subcharacteristic condition *with respect to the relaxation system* (1.1).

Chen et al. [7] proved that the subcharacteristic condition is satisfied if there exists a convex entropy function for the relaxation system (1.1), and that this entropy is locally dissipated by the relaxation term $\boldsymbol{R}$.

1.2. The model hierarchy. In a completely general (averaged) two-phase flow model, one may imagine that the two phases have separate pressures p_k, temperatures T_k, chemical potentials μ_k[1] and velocities v_k, where k is the phase index. The system can then be moved towards equilibrium by employing *relaxation source terms*, causing volume transfer due to pressure differences, heat transfer due to temperature differences, mass transfer due to chemical potential differences, and momentum transfer due to velocity differences between the two phases.

In our paper, we consider only homogeneous flow models, i.e. models where the phase velocities are equal. Discussion of models with different velocities, typically called two-fluid models, may be found in Refs. [1, 9, 22, 30]. We are then left with three relaxation processes, namely relaxation of pressure, temperature and chemical potential. By considering either the equilibrium (stiff) limit or the non-equilibrium (non-stiff) limit of these three processes, we get a hierarchy of models with different equilibrium assumptions.

Figure 1.1 illustrates this hierarchy, where circles symbolise models and arrows denote how the models are related through equilibrium assumptions on individual variables. Each arrow corresponds to a subcharacteristic condition for the wave speeds of the two models which the arrow connects. To the far left in this figure, we find the basic model, denoted 0, and to the far right, we find the homogeneous equilibrium model ($pT\mu$), in which the two phases are in full equilibrium. The full hierarchy is based on the work by Flåtten and Lund [10], who developed the basis (the basic model) for the hierarchy, along with the p, pT, $p\mu$ and $pT\mu$-models, shown

[1] Not to be confused with dynamic viscosity.

with dashed lines in Fig. 1.1. In the present work, we complete the hierarchy with the T, μ and $T\mu$-models, and the seven related subcharacteristic conditions, shown with solid lines in Fig. 1.1.

In this paper, we will present each of the models in this hierarchy. In particular, the formulation of the hyperbolic relaxation systems and the wave velocities (and hence the speed of sound) of the models will be presented, and we will explicitly show how the subcharacteristic condition is satisfied for each equilibrium assumption. More specifically, we will show how to relate the mixture speed of sound $\tilde{a}$ of an equilibrium model $\mathcal{X}$ and the corresponding relaxation (non-equilibrium) model $\mathcal{Y}$ by writing

$$\tilde{a}_{\mathcal{X}}^{-2} = \tilde{a}_{\mathcal{Y}}^{-2} + Z_{\mathcal{X}}^{\mathcal{Y}}, \tag{1.10}$$

where $Z_{\mathcal{X}}^{\mathcal{Y}}$ is a positive term expressed using sums of squares. This is shown to be sufficient to satisfy the subcharacteristic condition of Definition 1.

Stiff relaxation terms will cause dispersion of sound waves, with a speed of sound dependent on the wave number and the relaxation parameter ε. For more discussion regarding sound wave dispersion in certain models, see e.g. Städtke [26, Chap. 6] or Jinliang and Tingkuan [14]. We will focus our analysis on the non-stiff limit and the equilibrium limit, which are without dispersion.

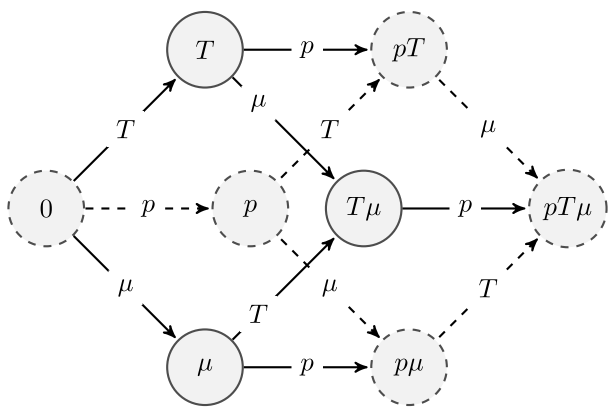


Figure 1.1: Model hierarchy. Each circle symbolises a two-phase flow model assuming equilibrium in zero or more of the variables p (pressure), T (temperature) and μ (chemical potential). Arrows represent a relaxation process of one variable, pointing in the direction of equilibrium in that variable. Solid lines indicate original contributions in the present paper, dashed lines indicate results presented in Ref. [10].

1.3. Paper outline. In the following, we will in turn present each of the eight different models shown in Figure 1.1, in Sections 2–9. Three of the models have, to the author's knowledge, not been described elsewhere, and thus represent original contributions. The models in question are the thermal equilibrium, the chemical equilibrium and the thermal-chemical equilibrium models, described in Sections 4, 5 and 8, respectively. The remaining models are the ones developed by Flåtten and Lund [10], which are all briefly included here for completeness. For each model, we aim towards an explicit expression of the mixture speed of sound, and prove that the subcharacteristic condition of Definition 1 is satisfied by relating speeds of sound in the different models using sums of squares.

In Section 10, we show plots of the mixture speeds of sound in the models of the hierarchy as functions of gas volume fraction, for relevant cases for water and carbon dioxide. Finally, Section 11 draws some conclusions and outlines possible further work.

2. Basic model. In this section, we present the basic one-dimensional two-phase flow model, in which we let the two phases have separate pressures, temperatures and chemical potentials, while the velocity v is equal in the two phases. Heat, mass and volume transfer between the phases are modelled using relaxation source terms. The model was proposed in this form by Flåtten and Lund [10], and forms the basis from which we can derive the other models in the hierarchy.

2.1. Mass balance. In general, we have one mass balance equation for each phase, which may be written as [10]

$$\frac{\partial(\alpha_{\mathrm{g}}\rho_{\mathrm{g}})}{\partial t}+\frac{\partial(\alpha_{\mathrm{g}}\rho_{\mathrm{g}}v)}{\partial x}=\mathcal{K}(\mu_\ell-\mu_{\mathrm{g}}), \tag{2.1}$$

$$\frac{\partial(\alpha_\ell\rho_\ell)}{\partial t}+\frac{\partial(\alpha_\ell\rho_\ell v)}{\partial x}=\mathcal{K}(\mu_{\mathrm{g}}-\mu_\ell), \tag{2.2}$$

where we use the following notation:

α_k	volume fraction of phase k
ρ_k	density of phase k
v	fluid velocity
μ_k	chemical potential of phase k
$\mathcal{K}\geq 0$	chemical potential relaxation parameter

Here the chemical potential relaxation source term ensures that mass flows from high to low chemical potential, if we only assume that $\mathcal{K}\geq 0$. Mass transfer modelled using such a relaxation term can be found in the works of e.g. Saurel et al. [23] and Stewart and Wendroff [25]. Adding the two equations (2.1)–(2.2) yields the conservation equation for total mass,

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho v)}{\partial x}=0. \tag{2.3}$$

Here, the mixture density ρ is given by

$$\rho=\alpha_{\mathrm{g}}\rho_{\mathrm{g}}+\alpha_\ell\rho_\ell. \tag{2.4}$$

2.2. Volume advection. We assume that volume transfer, in Lagrangian coordinates, can only be caused by differences in pressure, which is a common assumption also found e.g. in models by Baer and Nunziato [1] and Saurel et al. [22],

$$\mathrm{D}_t\alpha_{\mathrm{g}}=\mathcal{J}(p_{\mathrm{g}}-p_\ell), \tag{2.5}$$

where we have introduced the *material derivative*, defined by

$$\mathrm{D}_t\equiv\frac{\partial}{\partial t}+v\frac{\partial}{\partial x}, \tag{2.6}$$

and the notation

p_k	pressure of phase k
$\mathcal{J}\geq 0$	pressure relaxation parameter

Here, we note that the pressure relaxation causes volume to be transferred *to* the phase with highest pressure, i.e. the expanding phase has the highest pressure. The only assumption made is that the relaxation parameter is non-negative, $\mathcal{J}\geq 0$.

2.3. Momentum conservation. Since the basic model is defined as a homogeneous flow model, with equal velocity v for the two phases, the momentum conservation may be formulated as a conservation equation for the total momentum,

$$\frac{\partial(\rho v)}{\partial t}+\frac{\partial(\rho v^2+\alpha_{\mathrm{g}}p_{\mathrm{g}}+\alpha_\ell p_\ell)}{\partial x}=0. \tag{2.7}$$

2.4. Energy equations. We assume that each relaxation process should conserve energy and that in Lagrangian coordinates, only the relaxation terms contribute to entropy changes. This allows us to derive energy equations for each phase, which may be written as [10]

$$\begin{aligned}\frac{\partial E_{\mathrm{g}}}{\partial t}+\frac{\partial (vE_{\mathrm{g}})}{\partial x}+\alpha_{\mathrm{g}}p_{\mathrm{g}}\frac{\partial v}{\partial x}+\frac{v}{\rho}m_{\mathrm{g}}\frac{\partial(\alpha_{\mathrm{g}}p_{\mathrm{g}}+\alpha_\ell p_\ell)}{\partial x}&\\ =\mathcal{H}(T_\ell-T_{\mathrm{g}})+p^*\mathcal{J}(p_\ell-p_{\mathrm{g}})+\left(\mu^*+\frac{1}{2}v^2\right)\mathcal{K}(\mu_\ell-\mu_{\mathrm{g}}),&\end{aligned} \tag{2.8}$$

$$\begin{aligned}\frac{\partial E_\ell}{\partial t}+\frac{\partial (vE_\ell)}{\partial x}+\alpha_\ell p_\ell\frac{\partial v}{\partial x}+\frac{v}{\rho}m_\ell\frac{\partial(\alpha_{\mathrm{g}}p_{\mathrm{g}}+\alpha_\ell p_\ell)}{\partial x}&\\ =\mathcal{H}(T_{\mathrm{g}}-T_\ell)+p^*\mathcal{J}(p_{\mathrm{g}}-p_\ell)+\left(\mu^*+\frac{1}{2}v^2\right)\mathcal{K}(\mu_{\mathrm{g}}-\mu_\ell),&\end{aligned} \tag{2.9}$$

where p^* and μ^* are the pressure and chemical potential, respectively, at the gas-liquid interface. The detailed derivation can be found in Ref. [10]. For brevity, we have also introduced $m_k=\alpha_k\rho_k$, the mass per volume of phase k. The total energy in each phase, E_k, is given by

$$E_k=\alpha_k\rho_k(e_k+\frac{1}{2}v^2). \tag{2.10}$$

The temperature relaxation parameter is denoted $\mathcal{H}\geq 0$, and the corresponding heat source term $\mathcal{H}(T_\ell-T_{\mathrm{g}})$ causes heat to flow from the hot to the cold phase.

2.5. Entropy evolution. When deriving the wave velocities of the present model and other models in the hierarchy, it is often useful to formulate the model using entropy evolution equations instead of the energy equations (2.8)–(2.9). These can be formulated as [10]

$$\mathrm{D}_t s_{\mathrm{g}}=\left(\frac{\mu^*-\mu_{\mathrm{g}}}{T_{\mathrm{g}}}-s_{\mathrm{g}}\right)\frac{\mathcal{K}}{m_{\mathrm{g}}}(\mu_\ell-\mu_{\mathrm{g}})+\frac{\mathcal{H}}{m_{\mathrm{g}}}\frac{T_\ell-T_{\mathrm{g}}}{T_{\mathrm{g}}}+\frac{p^*-p_{\mathrm{g}}}{m_{\mathrm{g}}T_{\mathrm{g}}}\mathcal{J}(p_\ell-p_{\mathrm{g}}), \tag{2.11}$$

$$\mathrm{D}_t s_\ell=\left(\frac{\mu^*-\mu_\ell}{T_\ell}-s_\ell\right)\frac{\mathcal{K}}{m_\ell}(\mu_{\mathrm{g}}-\mu_\ell)+\frac{\mathcal{H}}{m_\ell}\frac{T_{\mathrm{g}}-T_\ell}{T_\ell}+\frac{p^*-p_\ell}{m_\ell T_\ell}\mathcal{J}(p_{\mathrm{g}}-p_\ell), \tag{2.12}$$

where s_k is the entropy density of phase k. These equations may also be formulated in a balance form,

$$T_{\mathrm{g}}\left(\frac{\partial(m_{\mathrm{g}}s_{\mathrm{g}})}{\partial t}+\frac{\partial(m_{\mathrm{g}}s_{\mathrm{g}}v)}{\partial x}\right)=\mathcal{H}(T_\ell-T_{\mathrm{g}})+(p^*-p_{\mathrm{g}})\mathcal{J}(p_\ell-p_{\mathrm{g}})+(\mu^*-\mu_{\mathrm{g}})\mathcal{K}(\mu_\ell-\mu_{\mathrm{g}}), \tag{2.13}$$

$$T_\ell\left(\frac{\partial(m_\ell s_\ell)}{\partial t}+\frac{\partial(m_\ell s_\ell v)}{\partial x}\right)=\mathcal{H}(T_{\mathrm{g}}-T_\ell)+(p^*-p_\ell)\mathcal{J}(p_{\mathrm{g}}-p_\ell)+(\mu^*-\mu_\ell)\mathcal{K}(\mu_{\mathrm{g}}-\mu_\ell). \tag{2.14}$$

The latter equations may be derived by using the entropy equations (2.11)–(2.12), the mass balance equations (2.1)–(2.2) and the volume fraction equation (2.5).

2.6. The laws of thermodynamics. An important point made by Flåtten and Lund [10] is that this basic model satisfies the first and second law of thermodynamics, which is a sensible requirement to have on any two-phase flow model. By adding the two energy equations (2.8)–(2.9), we get

$$\frac{\partial(E_{\mathrm{g}}+E_\ell)}{\partial t}+\frac{\partial[(E_{\mathrm{g}}+E_\ell+\alpha_{\mathrm{g}}p_{\mathrm{g}}+\alpha_\ell p_\ell)v]}{\partial x}=0, \tag{2.15}$$

thus the total energy is conserved, and the model fulfils the first law. The second law, expressing that entropy should be non-decreasing, is also satisfied, only requiring that

$$\mathcal{H} \geq 0, \tag{2.16}$$

$$\mathcal{J} \geq 0, \tag{2.17}$$

$$\mathcal{K} \geq 0, \tag{2.18}$$

$$\min(p_{\mathrm{g}}, p_\ell) \leq p^* \leq \max(p_{\mathrm{g}}, p_\ell), \tag{2.19}$$

$$\min(\mu_{\mathrm{g}}, \mu_\ell) \leq \mu^* \leq \max(\mu_{\mathrm{g}}, \mu_\ell). \tag{2.20}$$

The full proof can be found in Ref. [10].

2.7. Wave velocities. In the non-stiff limit $\mathcal{K}, \mathcal{J}, \mathcal{H} \to 0$, the wave velocities of the basic model Eqs. (2.1)–(2.2),(2.5)–(2.7),(2.11)–(2.12) can be found to be [10]

$$\boldsymbol{\lambda}_0 = \{v - \tilde{a}_0, v, v, v, v, v + \tilde{a}_0\}, \tag{2.21}$$

where $\tilde{a}_0$ is the mixture speed of sound of the basic model, given by

$$\tilde{a}_0 = \frac{m_{\mathrm{g}} c_{\mathrm{g}}^2 + m_\ell c_\ell^2}{\rho}, \tag{2.22}$$

i.e. a mass weighted average of the single-phase speeds of sound, which in turn (for phase k) are defined as

$$c_k^2 = \left(\frac{\partial p_k}{\partial \rho_k}\right)_{s_k}. \tag{2.23}$$

3. Pressure relaxation. In this section, we consider the model that results when we impose volume transfer equilibrium in the basic model of Section 2. In other words, we let the pressure relaxation parameter $\mathcal{J}$ go to infinity, which we expect to correspond to the assumption

$$p_{\mathrm{g}} = p_\ell = p^* = p, \tag{3.1}$$

i.e. mechanical equilibrium between the two phases. The mechanical equilibrium model equations may be obtained by replacing the pressure relaxation term $\mathcal{J}(p_{\mathrm{g}} - p_\ell)$ using the volume fraction equation (2.5), as described in detail by Flåtten et al. [11]. The full model equations are not stated here, but the derivation may be found in Ref. [10]. This five-equation model has been studied by a number of authors [11, 15, 20, 23, 24, 26], with slightly varying formulations.

3.1. Wave velocities. The wave velocities of the mechanical equilibrium model, in the non-stiff limit where $\mathcal{H}, \mathcal{K} \to 0$, are given by [11]

$$\boldsymbol{\lambda}_p = \{v - \tilde{a}_p, v, v, v, v + \tilde{a}_p\}, \tag{3.2}$$

where $\tilde{a}_p$ is the mixture speed of sound, given by

$$\tilde{a}_p^{-2} = \rho\left(\frac{\alpha_{\mathrm{g}}}{\rho_{\mathrm{g}} c_{\mathrm{g}}^2} + \frac{\alpha_\ell}{\rho_\ell c_\ell^2}\right). \tag{3.3}$$

This is a classic, well-known expression, also referred to as the *Wood speed of sound* [24] or *Wallis speed of sound* [27].

As shown by Flåtten and Lund [10], the mechanical equilibrium model satisfies the subcharacteristic condition with respect to the basic model, only requiring $\rho_k > 0$. This can be shown by writing the mixture speed of sound as

$$\tilde{a}_p^{-2} = \tilde{a}_0^{-2} + Z_p^0, \tag{3.4}$$

where

$$Z_p^0 = \tilde{a}_0^{-2} \frac{\alpha_{\mathrm{g}} \alpha_\ell}{\rho_{\mathrm{g}} c_{\mathrm{g}}^2 \rho_\ell c_\ell^2} (\rho_{\mathrm{g}} c_{\mathrm{g}}^2 - \rho_\ell c_\ell^2)^2. \tag{3.5}$$

4. Temperature relaxation. In this section, we consider the model that results when we impose heat transfer equilibrium in the basic model of Section 2. In other words, we let the temperature relaxation parameter $\mathcal{H}$ go to infinity, which we expect to correspond to the assumption

$$T_{\mathrm{g}} = T_\ell = T, \tag{4.1}$$

i.e. thermal equilibrium between the two phases. The model equations and wave velocities for this model have not been found elsewhere, and will thus be derived here.

When we let the temperature relaxation parameter go to infinity, $\mathcal{H} \to \infty$, the value of the temperature relaxation term $\mathcal{H}(T_\ell - T_{\mathrm{g}})$ is no longer defined. Thus, to derive the equations describing the current model, we find it necessary to determine an explicit expression for the temperature relaxation (or heat transfer) term.

To this end, we consider the two following thermodynamic differentials:

$$\mathrm{d}T = \frac{\Gamma_{\mathrm{g}} T}{\rho_{\mathrm{g}} c_{\mathrm{g}}^2} \mathrm{d}p_{\mathrm{g}} + \frac{T}{c_{p,\mathrm{g}}} \mathrm{d}s_{\mathrm{g}} = \frac{\Gamma_\ell T}{\rho_\ell c_\ell^2} \mathrm{d}p_\ell + \frac{T}{c_{p,\ell}} \mathrm{d}s_\ell, \tag{4.2}$$

$$\mathrm{d}p_k = c_k^2 \mathrm{d}\rho_k + \rho_k \Gamma_k T \mathrm{d}s_k, \tag{4.3}$$

where Γ_k is the *Grüneisen coefficient* and $c_{p,k}$ is the specific heat capacity at constant pressure, defined by

$$\Gamma_k = \frac{1}{\rho_k} \left(\frac{\partial p_k}{\partial e_k} \right)_{\rho_k}, \tag{4.4}$$

$$c_{p,k} = T_k \left(\frac{\partial s_k}{\partial T_k} \right)_{p_k}. \tag{4.5}$$

By using Eqs. (2.1)–(2.2), (2.5), (2.11)–(2.12), together with Eqs. (4.2)–(4.3) expressed with the material derivative, we may solve for the heat transfer term, which yields

$$\begin{aligned} \mathcal{H}(T_\ell - T_{\mathrm{g}}) = {} & \frac{\Gamma_{\mathrm{g}} - \Gamma_\ell}{\frac{\Gamma_{\mathrm{g}}^2}{m_{\mathrm{g}} c_{\mathrm{g}}^2} + \frac{1}{C_{p,\mathrm{g}} T} + \frac{\Gamma_\ell^2}{m_\ell c_\ell^2} + \frac{1}{C_{p,\ell} T}} \frac{\partial v}{\partial x} \\ & - \frac{\frac{\Gamma_{\mathrm{g}}}{m_{\mathrm{g}}} + \frac{\Gamma_\ell}{m_\ell} + \left(\frac{\Gamma_{\mathrm{g}}^2}{m_{\mathrm{g}} c_{\mathrm{g}}^2} + \frac{1}{C_{p,\mathrm{g}} T} \right) (\mu^* - h_{\mathrm{g}}) + \left(\frac{\Gamma_\ell^2}{m_\ell c_\ell^2} + \frac{1}{C_{p,\ell} T} \right) (\mu^* - h_\ell)}{\frac{\Gamma_{\mathrm{g}}^2}{m_{\mathrm{g}} c_{\mathrm{g}}^2} + \frac{1}{C_{p,\mathrm{g}} T} + \frac{\Gamma_\ell^2}{m_\ell c_\ell^2} + \frac{1}{C_{p,\ell} T}} \mathcal{K}(\mu_\ell - \mu_{\mathrm{g}}) \\ & - \frac{\frac{\Gamma_{\mathrm{g}}}{\alpha_{\mathrm{g}}} + \frac{\Gamma_\ell}{\alpha_\ell} + \left(\frac{\Gamma_{\mathrm{g}}^2}{m_{\mathrm{g}} c_{\mathrm{g}}^2} + \frac{1}{C_{p,\mathrm{g}} T} \right) (p^* - p_{\mathrm{g}}) + \left(\frac{\Gamma_\ell^2}{m_\ell c_\ell^2} + \frac{1}{C_{p,\ell} T} \right) (p^* - p_\ell)}{\frac{\Gamma_{\mathrm{g}}^2}{m_{\mathrm{g}} c_{\mathrm{g}}^2} + \frac{1}{C_{p,\mathrm{g}} T} + \frac{\Gamma_\ell^2}{m_\ell c_\ell^2} + \frac{1}{C_{p,\ell} T}} \mathcal{J}(p_\ell - p_{\mathrm{g}}), \end{aligned} \tag{4.6}$$

where

$$C_{p,k} = \alpha_k \rho_k c_{p,k} \tag{4.7}$$

is the extensive heat capacity at constant pressure. We may now formulate the equations describing the thermal equilibrium model.

4.1. The thermal equilibrium model. The thermal equilibrium model can now be summarised using the following equations.

- Mass balance:

$$\frac{\partial(\alpha_{\mathrm{g}} \rho_{\mathrm{g}})}{\partial x} + \frac{\partial(\alpha_{\mathrm{g}} \rho_{\mathrm{g}} v)}{\partial x} = \mathcal{K}(\mu_\ell - \mu_{\mathrm{g}}), \tag{4.8}$$

$$\frac{\partial(\alpha_\ell \rho_\ell)}{\partial x} + \frac{\partial(\alpha_\ell \rho_\ell v)}{\partial x} = \mathcal{K}(\mu_{\mathrm{g}} - \mu_\ell), \tag{4.9}$$

- Momentum conservation:

$$\frac{\partial(\rho v)}{\partial t}+\frac{\partial(\rho v^2+\alpha_{\mathrm{g}}p_{\mathrm{g}}+\alpha_\ell p_\ell)}{\partial x}=0, \tag{4.10}$$

- Volume fraction evolution:

$$\mathrm{D}_t\alpha_{\mathrm{g}}=\mathcal{J}(p_{\mathrm{g}}-p_\ell), \tag{4.11}$$

- Energy conservation:

$$\frac{\partial E}{\partial t}+\frac{\partial[(E+p)v]}{\partial x}=0. \tag{4.12}$$

These model equations are Eqs. (2.1)–(2.2), (2.5), (2.7) and (2.15) from the basic model. Herein, E is the total energy per volume, defined by

$$E=E_{\mathrm{g}}+E_\ell\equiv\alpha_{\mathrm{g}}\rho_{\mathrm{g}}(e_{\mathrm{g}}+\frac{1}{2}v^2)+\alpha_\ell\rho_\ell(e_\ell+\frac{1}{2}v^2). \tag{4.13}$$

4.2. Wave velocities. We now wish to derive the wave velocities in the non-stiff limit where the pressure and chemical potential relaxation parameters vanish, $\mathcal{J},\mathcal{K}\to 0$. To this end, we find it useful to derive the material derivative of the effective pressure $p_{\mathrm{eff}}\equiv\alpha_{\mathrm{g}}p_{\mathrm{g}}+\alpha_\ell p_\ell$,

$$\mathrm{D}_tp_{\mathrm{eff}}=\alpha_{\mathrm{g}}\mathrm{D}_tp_{\mathrm{g}}+\alpha_\ell\mathrm{D}_tp_\ell+(p_{\mathrm{g}}-p_\ell)\mathrm{D}_t\alpha_{\mathrm{g}}. \tag{4.14}$$

We insert for the pressure differentials D_tp_k from Eq. (4.3), and then rewrite the density differentials $\mathrm{D}_t\rho_k$ using the product rule on D_tm_k, yielding

$$\mathrm{D}_tp_{\mathrm{eff}}=c_{\mathrm{g}}^2\mathrm{D}_tm_{\mathrm{g}}+m_{\mathrm{g}}\Gamma_{\mathrm{g}}T\mathrm{D}_ts_{\mathrm{g}}+c_\ell^2\mathrm{D}_tm_\ell+m_\ell\Gamma_\ell T\mathrm{D}_ts_\ell, \tag{4.15}$$

where have used that $\mathrm{D}_t\alpha_{\mathrm{g}}\to 0$ since $\mathcal{J}\to 0$. The terms D_tm_k may be found by rewriting the mass balance equations (4.8)–(4.9). We also replace D_ts_k from Eqs. (2.11)–(2.12) and (4.6), keeping in mind that $\mathcal{K},\mathcal{J}\to 0$, and finally get

$$\mathrm{D}_tp_{\mathrm{eff}}=-\rho\tilde{a}_T^2\frac{\partial v}{\partial x}, \tag{4.16}$$

where

$$\tilde{a}_T^2=\frac{1}{\rho}\frac{m_\ell c_\ell^2m_{\mathrm{g}}c_{\mathrm{g}}^2\left(\frac{\Gamma_{\mathrm{g}}}{m_{\mathrm{g}}c_{\mathrm{g}}^2}+\frac{\Gamma_\ell}{m_\ell c_\ell^2}\right)^2+\frac{1}{T}\left(\frac{1}{C_{p,\mathrm{g}}}+\frac{1}{C_{p,\ell}}\right)(m_{\mathrm{g}}c_{\mathrm{g}}^2+m_\ell c_\ell^2)}{\frac{1}{m_{\mathrm{g}}c_{\mathrm{g}}^2}\Gamma_{\mathrm{g}}^2+\frac{1}{m_\ell c_\ell^2}\Gamma_\ell^2+\frac{1}{T}\left(\frac{1}{C_{p,\mathrm{g}}}+\frac{1}{C_{p,\ell}}\right)}. \tag{4.17}$$

Using the gas mass balance equation (4.8) and total continuity equation (2.3), we find that the gas mass fraction $Y_{\mathrm{g}}\equiv\frac{m_{\mathrm{g}}}{\rho}$ satisfies

$$\mathrm{D}_tY_{\mathrm{g}}=\frac{\mathcal{K}}{\rho}(\mu_\ell-\mu_{\mathrm{g}}). \tag{4.18}$$

Thus, in the non-stiff limit $\mathcal{J},\mathcal{K}\to 0$, we know from Eqs. (2.5) and (4.18) that Y_{g} and α_{g} are characteristic variables with a corresponding eigenvalue v. The remaining model equations, namely the total continuity equation (2.3), momentum conservation (4.10) and pressure evolution equation (4.16) may be formulated as a quasi-linear system,

$$\boldsymbol{u}_t+\begin{bmatrix}0&1&0\\-v^2&2v&1\\-v\tilde{a}_T^2&\tilde{a}_T^2&v\end{bmatrix}\boldsymbol{u}_x\equiv\boldsymbol{u}_t+\boldsymbol{A}(\boldsymbol{u})\boldsymbol{u}_x=0, \tag{4.19}$$

where $\boldsymbol{u}=[\rho,\rho v,p_{\mathrm{eff}}]$. The eigenvalues of the matrix $\boldsymbol{A}(\boldsymbol{u})$ are given by $\{v-\tilde{a}_T,v,v+\tilde{a}_T\}$, so the eigenstructure of the full model is given by

$$\boldsymbol{\lambda}_T=\{v-\tilde{a}_T,v,v,v,v+\tilde{a}_T\}, \tag{4.20}$$

where the mixture speed of sound is $\tilde{a}_T$, given by Eq. (4.17).

4.2.1. The subcharacteristic condition with respect to the basic model. From Eqs. (2.22) and (4.17), we find that the mixture speed of sound of the thermal equilibrium model can be written as

$$\tilde{a}_T^{-2} = \tilde{a}_0^{-2} + Z_T^0, \tag{4.21}$$

where

$$Z_T^0 = \frac{1}{\tilde{a}_0^2} \frac{(\Gamma_\mathrm{g} - \Gamma_\ell)^2}{m_\ell c_\ell^2 m_\mathrm{g} c_\mathrm{g}^2 \left(\frac{\Gamma_\mathrm{g}}{m_\mathrm{g} c_\mathrm{g}^2} + \frac{\Gamma_\ell}{m_\ell c_\ell^2}\right)^2 + \left(\frac{1}{C_{p,\ell} T} + \frac{1}{C_{p,\mathrm{g}} T}\right) \rho \tilde{a}_0^2}. \tag{4.22}$$

Proposition 1. *The thermal equilibrium model given by Eqs.* (4.8)–(4.12) *satisfies the subcharacteristic condition with respect to the basic model of Section 2, subject only to the physically fundamental conditions*

$$\rho_k > 0,$$
$$c_{p,k} > 0,$$
$$T > 0.$$

Proof. By Eqs. (2.21) and (4.20), we see that the interlacing condition in Definition 1 reduces to the requirement that

$$\tilde{a}_0 \geq \tilde{a}_T, \tag{4.23}$$

which follows from Eqs. (4.21)–(4.22) and the given conditions for ρ_k, $c_{p,k}$ and T. □

5. Chemical potential relaxation. In this section, we investigate the model that arises when we impose mass transfer equilibrium in the basic model of Section 2. In other words, the phase transition between liquid and gas will be infinitely fast. This is equivalent to letting the chemical potential relaxation parameter $\mathcal{K}$ go to infinity, which we expect to correspond to the assumption

$$\mu_\mathrm{g} = \mu_\ell = \mu = \mu^*, \tag{5.1}$$

i.e. equal chemical potentials and chemical equilibrium. The model equations and wave velocities for this model have not been found elsewhere, and will thus be derived here.

5.1. Mass fraction evolution equations. In the limit $\mathcal{K} \to \infty$, the chemical potentials in the two phases are equal, $\mu_\mathrm{g} = \mu_\ell$, hence the value of the mass relaxation term $\mathcal{K}(\mu_\mathrm{g} - \mu_\ell)$ is undefined. To find an expression for this quantity, we find it necessary to derive some differentials. Since the chemical potentials are equal, $\mu_\mathrm{g} = \mu_\ell$, so are their differentials, $\mathrm{d}\mu_\mathrm{g} = \mathrm{d}\mu_\ell$, which yields

$$\frac{1}{\rho_\ell}\mathrm{d}p_\ell - s_\ell \mathrm{d}T_\ell = \frac{1}{\rho_\mathrm{g}}\mathrm{d}p_\mathrm{g} - s_\mathrm{g}\mathrm{d}T_\mathrm{g}. \tag{5.2}$$

The temperature and pressure differentials can be written as

$$\mathrm{d}T_k = \frac{\Gamma_k T_k}{\rho_k c_k^2}\mathrm{d}p_k + \frac{T_k}{c_{p,k}}\mathrm{d}s_k, \tag{5.3}$$

$$\mathrm{d}p_k = c_k^2 \mathrm{d}\rho_k + \rho_k \Gamma_k T_k \mathrm{d}s_k. \tag{5.4}$$

We then insert for the temperature differential (5.3) and then the pressure differential (5.4) in Eq. (5.2), which yields

$$\frac{\xi_\ell^2}{\rho_\ell}\mathrm{d}\rho_\ell + \left(\frac{\rho_\ell}{s_\ell}(c_\ell^2 - \xi_\ell^2) - \frac{s_\ell T_\ell}{c_{p,\ell}}\right)\mathrm{d}s_\ell = \frac{\xi_\mathrm{g}^2}{\rho_\mathrm{g}}\mathrm{d}\rho_\mathrm{g} + \left(\frac{\rho_\mathrm{g}}{s_\mathrm{g}}(c_\mathrm{g}^2 - \xi_\mathrm{g}^2) - \frac{s_\mathrm{g} T_\mathrm{g}}{c_{p,\mathrm{g}}}\right)\mathrm{d}s_\mathrm{g}, \tag{5.5}$$

where we have introduced the abbreviation $\xi_k^2 \equiv c_k^2 - \Gamma_k s_k T_k$. Next, we have use for the differential of the total density,

$$\mathrm{d}\rho = \alpha_\mathrm{g}\mathrm{d}\rho_\mathrm{g} + \alpha_\ell \mathrm{d}\rho_\ell + (\rho_\mathrm{g} - \rho_\ell)\mathrm{d}\alpha_\mathrm{g}, \tag{5.6}$$

and gas mass fraction differential

$$\mathrm{d}Y_\mathrm{g} = -\frac{m_\mathrm{g}}{\rho^2}\mathrm{d}\rho + \frac{1}{\rho}(\alpha_\mathrm{g}\mathrm{d}\rho_\mathrm{g} + \rho_\mathrm{g}\mathrm{d}\alpha_\mathrm{g}). \tag{5.7}$$

By writing Eqs. (5.5)–(5.7) using the material derivative, together with the equations for entropy (2.11)–(2.12), volume fraction (2.5), total continuity (2.3) and gas mass fraction (4.18), we arrive at the mass fraction evolution equation,

$$\begin{aligned} \mathrm{D}_t Y_\mathrm{g} = {} & \frac{1}{\left(\frac{\xi_\mathrm{g}^4}{m_\mathrm{g}c_\mathrm{g}^2} + \frac{\xi_\ell^4}{m_\ell c_\ell^2} + \frac{s_\ell^2 T_\ell}{C_{p,\ell}} + \frac{s_\mathrm{g}^2 T_\mathrm{g}}{C_{p,\mathrm{g}}}\right)\rho} \\ & \cdot \Bigg[(\xi_\mathrm{g}^2 - \xi_\ell^2)\frac{\partial v}{\partial x} + \left(-\frac{\xi_\ell^2(c_\ell^2 - \xi_\ell^2)}{m_\ell c_\ell^2 s_\ell T_\ell} - \frac{\xi_\mathrm{g}^2(c_\mathrm{g}^2 - \xi_\mathrm{g}^2)}{m_\mathrm{g}c_\mathrm{g}^2 s_\mathrm{g} T_\mathrm{g}} + \frac{s_\ell}{C_{p,\ell}} + \frac{s_\mathrm{g}}{C_{p,\mathrm{g}}}\right)\mathcal{H}(T_\ell - T_\mathrm{g}) \\ & + \left(p_\mathrm{g}^*\left(\frac{s_\mathrm{g}}{C_{p,\mathrm{g}}} + \frac{\xi_\mathrm{g}^2(\xi_\mathrm{g}^2 - c_\mathrm{g}^2)}{m_\mathrm{g}c_\mathrm{g}^2 s_\mathrm{g} T_\mathrm{g}}\right) + p_\ell^*\left(\frac{s_\ell}{C_{p,\ell}} + \frac{\xi_\ell^2(\xi_\ell^2 - c_\ell^2)}{m_\ell c_\ell^2 s_\ell T_\ell}\right) - \frac{\xi_\mathrm{g}^2}{\alpha_\mathrm{g}} - \frac{\xi_\ell^2}{\alpha_\ell}\right)\mathcal{J}(p_\ell - p_\mathrm{g})\Bigg], \end{aligned} \tag{5.8}$$

where we have introduced an interface-bulk pressure difference $p_k^* = p^* - p_k$.

5.2. The chemical equilibrium model. The chemical equilibrium model may now be formulated using the following equations.

- Mass conservation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0, \tag{5.9}$$

- Momentum conservation:

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho v^2 + \alpha_\mathrm{g} p_\mathrm{g} + \alpha_\ell p_\ell)}{\partial x} = 0, \tag{5.10}$$

- Volume fraction evolution:

$$\mathrm{D}_t \alpha_\mathrm{g} = \mathcal{J}(p_\mathrm{g} - p_\ell), \tag{5.11}$$

- Energy equations:

$$\begin{aligned} & \frac{\partial E_\mathrm{g}}{\partial t} + \frac{\partial(vE_\mathrm{g})}{\partial x} + \frac{v}{\rho}m_\mathrm{g}\frac{\partial p_\mathrm{eff}}{\partial x} - \left(\frac{(\xi_\mathrm{g}^2 - \xi_\ell^2)\left(\mu + \frac{1}{2}v^2\right)}{\frac{\xi_\mathrm{g}^4}{m_\mathrm{g}c_\mathrm{g}^2} + \frac{\xi_\ell^4}{m_\ell c_\ell^2} + \frac{s_\ell^2 T_\ell}{C_{p,\ell}} + \frac{s_\mathrm{g}^2 T_\mathrm{g}}{C_{p,\mathrm{g}}}} - \alpha_\mathrm{g} p_\mathrm{g}\right)\frac{\partial v}{\partial x} \\ & = \left[\frac{\left(-\frac{\Gamma_\ell \xi_\ell^2}{m_\ell c_\ell^2} - \frac{\Gamma_\mathrm{g}\xi_\mathrm{g}^2}{m_\mathrm{g}c_\mathrm{g}^2} + \frac{s_\ell}{C_{p,\ell}} + \frac{s_\mathrm{g}}{C_{p,\mathrm{g}}}\right)\left(\mu + \frac{1}{2}v^2\right)}{\frac{\xi_\mathrm{g}^4}{m_\mathrm{g}c_\mathrm{g}^2} + \frac{\xi_\ell^4}{m_\ell c_\ell^2} + \frac{s_\ell^2 T_\ell}{C_{p,\ell}} + \frac{s_\mathrm{g}^2 T_\mathrm{g}}{C_{p,\mathrm{g}}}} + 1\right]\mathcal{H}(T_\ell - T_\mathrm{g}) \\ & + \left[\frac{\left(\left(-\frac{\Gamma_\mathrm{g}\xi_\mathrm{g}^2}{m_\mathrm{g}c_\mathrm{g}^2} + \frac{s_\mathrm{g}}{C_{p,\mathrm{g}}}\right)p_\mathrm{g}^* + \left(-\frac{\Gamma_\ell \xi_\ell^2}{m_\ell c_\ell^2} + \frac{s_\ell}{C_{p,\ell}}\right)p_\ell^* - \frac{\xi_\mathrm{g}^2}{\alpha_\mathrm{g}} - \frac{\xi_\ell^2}{\alpha_\ell}\right)\left(\mu + \frac{1}{2}v^2\right)}{\frac{\xi_\mathrm{g}^4}{m_\mathrm{g}c_\mathrm{g}^2} + \frac{\xi_\ell^4}{m_\ell c_\ell^2} + \frac{s_\ell^2 T_\ell}{C_{p,\ell}} + \frac{s_\mathrm{g}^2 T_\mathrm{g}}{C_{p,\mathrm{g}}}} + p^*\right]\mathcal{J}(p_\ell - p_\mathrm{g}), \end{aligned} \tag{5.12}$$

$$\frac{\partial E_\ell}{\partial t} + \frac{\partial (vE_\ell)}{\partial x} + \frac{v}{\rho} m_\ell \frac{\partial p_{\text{eff}}}{\partial x} - \left(\frac{(\xi_\ell^2 - \xi_{\text{g}}^2)\left(\mu + \frac{1}{2}v^2\right)}{\frac{\xi_\ell^4}{m_\ell c_\ell^2} + \frac{\xi_{\text{g}}^4}{m_{\text{g}} c_{\text{g}}^2} + \frac{s_{\text{g}}^2 T_{\text{g}}}{C_{p,\text{g}}} + \frac{s_\ell^2 T_\ell}{C_{p,\ell}}} - \alpha_\ell p_\ell \right) \frac{\partial v}{\partial x}$$
$$= \left[\frac{\left(-\frac{\Gamma_{\text{g}} \xi_{\text{g}}^2}{m_{\text{g}} c_{\text{g}}^2} - \frac{\Gamma_\ell \xi_\ell^2}{m_\ell c_\ell^2} + \frac{s_{\text{g}}}{C_{p,\text{g}}} + \frac{s_\ell}{C_{p,\ell}} \right)\left(\mu + \frac{1}{2}v^2\right)}{\frac{\xi_\ell^4}{m_\ell c_\ell^2} + \frac{\xi_{\text{g}}^4}{m_{\text{g}} c_{\text{g}}^2} + \frac{s_{\text{g}}^2 T_{\text{g}}}{C_{p,\text{g}}} + \frac{s_\ell^2 T_\ell}{C_{p,\ell}}} + 1 \right] \mathcal{H}(T_{\text{g}} - T_\ell)$$
$$+ \left[\frac{\left(\left(-\frac{\Gamma_\ell \xi_\ell^2}{m_\ell c_\ell^2} + \frac{s_\ell}{C_{p,\ell}} \right) p_\ell^* + \left(-\frac{\Gamma_{\text{g}} \xi_{\text{g}}^2}{m_{\text{g}} c_{\text{g}}^2} + \frac{s_{\text{g}}}{C_{p,\text{g}}} \right) p_{\text{g}}^* - \frac{\xi_\ell^2}{\alpha_\ell} - \frac{\xi_{\text{g}}^2}{\alpha_{\text{g}}} \right)\left(\mu + \frac{1}{2}v^2\right)}{\frac{\xi_\ell^4}{m_\ell c_\ell^2} + \frac{\xi_{\text{g}}^4}{m_{\text{g}} c_{\text{g}}^2} + \frac{s_{\text{g}}^2 T_{\text{g}}}{C_{p,\text{g}}} + \frac{s_\ell^2 T_\ell}{C_{p,\ell}}} + p^* \right] \mathcal{J}(p_{\text{g}} - p_\ell). \tag{5.13}$$

Herein, the continuity, momentum conservation and volume fraction equations are the ones known from the basic model, Eqs. (2.3), (2.5) and (2.7), while the energy equations (5.12)–(5.13) are derived by inserting for the chemical potential relaxation term $\mathcal{K}(\mu_\ell - \mu_{\text{g}})$ in Eqs. (2.8)–(2.9) using Eqs. (5.8) and (4.18).

5.3. Wave velocities. We wish to calculate the wave velocities, and hence the mixture speed of sound, of the chemical equilibrium model (5.9)–(5.13) in the non-stiff limit where $\mathcal{H}, \mathcal{J} \to 0$. To this end, we find it useful to derive an evolution equation for the effective pressure p_{eff}.

The material derivative of the effective pressure p_{eff} is given by Eq. (4.14). In this equation, we replace $\mathrm{D}_t p_{\text{g}}$ and $\mathrm{D}_t p_\ell$ using Eqs. (5.2)–(5.4) and (5.6). We then insert for $\mathrm{D}_t s_{\text{g}}$ and $\mathrm{D}_t s_\ell$ by replacing the chemical potential relaxation term in the basic model entropy equations (2.11) and (2.12) using Eqs. (5.8) and (4.18). Finally, using that $\mathrm{D}_t \alpha_{\text{g}} = 0$ due to Eq. (2.5) and the fact that $\mathcal{J}, \mathcal{H} \to 0$, gives

$$\mathrm{D}_t p_{\text{eff}} = \tilde{a}_\mu^2 \mathrm{D}_t \rho, \tag{5.14}$$

where

$$\tilde{a}_\mu^2 = \frac{\left(\frac{\xi_\ell^2}{m_\ell c_\ell^2} + \frac{\xi_{\text{g}}^2}{m_{\text{g}} c_{\text{g}}^2} \right)^2 m_{\text{g}} c_{\text{g}}^2 m_\ell c_\ell^2 + (m_{\text{g}} c_{\text{g}}^2 + m_\ell c_\ell^2)\left(\frac{s_\ell^2 T_\ell}{C_{p,\ell}} + \frac{s_{\text{g}}^2 T_{\text{g}}}{C_{p,\text{g}}} \right)}{\rho \left(\frac{\xi_{\text{g}}^4}{m_{\text{g}} c_{\text{g}}^2} + \frac{\xi_\ell^4}{m_\ell c_\ell^2} + \frac{s_\ell^2 T_\ell}{C_{p,\ell}} + \frac{s_{\text{g}}^2 T_{\text{g}}}{C_{p,\text{g}}} \right)}. \tag{5.15}$$

We may now write the full equation system in a quasi-linear form,

$$\boldsymbol{u}_t + \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -v^2 & 2v & 0 & 0 & 1 \\ vG & -G & v & 0 & 0 \\ vL & -L & 0 & v & 0 \\ -v\tilde{a}_\mu^2 & \tilde{a}_\mu^2 & 0 & 0 & v \end{bmatrix} \boldsymbol{u}_x \equiv \boldsymbol{u}_t + \boldsymbol{A}(\boldsymbol{u})\boldsymbol{u}_x = 0, \tag{5.16}$$

where $\boldsymbol{u} = [\rho, \rho v, s_{\text{g}}, s_\ell, p_{\text{eff}}]^{\mathrm{T}}$ and

$$G = \frac{s_{\text{g}}(\xi_\ell^2 - \xi_{\text{g}}^2)}{\rho c_{\text{g}}^2 m_{\text{g}} \left(\frac{\xi_\ell^4}{m_\ell c_\ell^2} + \frac{\xi_{\text{g}}^4}{m_{\text{g}} c_{\text{g}}^2} + \frac{s_\ell^2 T_\ell}{C_{p,\ell}} + \frac{s_{\text{g}}^2 T_{\text{g}}}{C_{p,\text{g}}} \right)}, \tag{5.17}$$

$$L = \frac{s_\ell(\xi_{\text{g}}^2 - \xi_\ell^2)}{\rho c_\ell^2 m_\ell \left(\frac{\xi_{\text{g}}^4}{m_{\text{g}} c_{\text{g}}^2} + \frac{\xi_\ell^4}{m_\ell c_\ell^2} + \frac{s_{\text{g}}^2 T_{\text{g}}}{C_{p,\text{g}}} + \frac{s_\ell^2 T_\ell}{C_{p,\ell}} \right)}. \tag{5.18}$$

The equation system has been formed by the equations for mass (5.9), momentum (5.10) and pressure (5.14), along with the entropy equations, which are obtained by replacing the mass transfer term in Eqs. (2.11)–(2.12) using Eqs. (4.18) and (5.8). The eigenvalues of the matrix $\boldsymbol{A}$ are

$$\boldsymbol{\lambda}_\mu \in \{v - \tilde{a}_\mu, v, v, v, v + \tilde{a}_\mu\}, \tag{5.19}$$

hence the mixture speed of sound of the chemical equilibrium model is $\tilde{a}_\mu$, given by Eq. (5.15).

5.3.1. The subcharacteristic condition with respect to the basic model. Using the expressions for the mixture speed of sound in the basic model (2.22) and the chemical equilibrium model (5.15), we can show that

$$\tilde{a}_{\mu}^{-2} = \tilde{a}_0^{-2} + Z_{\mu}^0, \tag{5.20}$$

where

$$Z_{\mu}^0 = \frac{(\xi_\ell^2 - \xi_{\mathrm{g}}^2)^2}{\left[\left(\frac{\xi_{\mathrm{g}}^2}{m_{\mathrm{g}} c_{\mathrm{g}}^2} + \frac{\xi_\ell^2}{m_\ell c_\ell^2}\right)^2 c_\ell^2 c_{\mathrm{g}}^2 m_{\mathrm{g}} m_\ell + \left(\frac{s_\ell^2 T_\ell}{C_{p,\ell}} + \frac{s_{\mathrm{g}}^2 T_{\mathrm{g}}}{C_{p,\mathrm{g}}}\right) \rho \tilde{a}_0^2\right] \tilde{a}_0^2}. \tag{5.21}$$

PROPOSITION 2. *The chemical equilibrium model given by Eqs.* (5.9)–(5.13) *satisfies the subcharacteristic condition with respect to the basic model of Section 2, subject only to the physically fundamental conditions*

$$\rho_k > 0,$$
$$c_{p,k} > 0,$$
$$T_k > 0.$$

Proof. From the eigenstructure of the basic model (2.21) and the chemical equilibrium model (5.19), we see that the interlacing condition in Definition 1 reduces to the requirement that

$$\tilde{a}_0 \geq \tilde{a}_{\mu}, \tag{5.22}$$

which follows from Eq. (5.20)–(5.21) and the given conditions for ρ_k, $c_{p,k}$ and T_k. □

6. Pressure-temperature relaxation. In this section, we investigate the model that arises when we impose volume and heat transfer equilibrium. In other words, we let the pressure and temperature relaxation parameters $\mathcal{J}, \mathcal{H}$ go to infinity. This corresponds to taking the limit

$$\mathcal{H} \to \infty \tag{6.1}$$

in the mechanical equilibrium model of Section 3, or equivalently taking the limit

$$\mathcal{J} \to \infty \tag{6.2}$$

in the thermal equilibrium model (4.8)–(4.12), which we expect to correspond to the assumptions

$$T_{\mathrm{g}} = T_\ell = T, \tag{6.3}$$
$$p_{\mathrm{g}} = p_\ell = p^* = p, \tag{6.4}$$

i.e. equal temperatures and pressures. The model equations may be found in Ref. [10].

6.1. Wave velocities. The wave structure of the mechanical-thermal equilibrium model was investigated by Flåtten et al. [11] in the general case of n different components with n mass balance equations, in the non-stiff limit where $\mathcal{K} \to 0$. In the case of two components, $n = 2$, the wave velocities were found to be

$$\boldsymbol{\lambda}_{pT} = \{v - \tilde{a}_{pT}, v, v, v + \tilde{a}_{pT}\}, \tag{6.5}$$

where

$$\tilde{a}_{pT}^{-2} = \rho\left(\frac{\alpha_{\mathrm{g}}}{\rho_{\mathrm{g}} c_{\mathrm{g}}^2} + \frac{\alpha_\ell}{\rho_\ell c_\ell^2}\right) + \rho T \frac{C_{p,\mathrm{g}} C_{p,\ell}}{C_{p,\mathrm{g}} + C_{p,\ell}} \left(\frac{\Gamma_\ell}{\rho_\ell c_\ell^2} - \frac{\Gamma_{\mathrm{g}}}{\rho_{\mathrm{g}} c_{\mathrm{g}}^2}\right)^2. \tag{6.6}$$

This model and its wave velocities are also described by Städtke [26, Chap. 4].

6.1.1. The subcharacteristic condition with respect to the p-model. As shown by Flåtten and Lund [10], the mechanical-thermal equilibrium model satisfies the subcharacteristic condition with respect to the mechanical equilibrium model of Section 3, given only the physically fundamental requirements $\rho_k > 0$, $c_{p,k} > 0$, $T > 0$. This is easily seen from Eq. (6.6),

$$\tilde{a}_{pT}^{-2} = \tilde{a}_p^{-2} + Z_{pT}^p, \tag{6.7}$$

where

$$Z_{pT}^p = \rho T \frac{C_{p,\mathrm{g}} C_{p,\ell}}{C_{p,\mathrm{g}} + C_{p,\ell}} \left(\frac{\Gamma_\ell}{\rho_\ell c_\ell^2} - \frac{\Gamma_\mathrm{g}}{\rho_\mathrm{g} c_\mathrm{g}^2} \right)^2. \tag{6.8}$$

6.1.2. The subcharacteristic condition with respect to the T-model. From Eqs. (4.17) and (6.6) we see that the mixture speed of sound in the mechanical-thermal equilibrium model may be expressed as

$$\tilde{a}_{pT}^{-2} = \tilde{a}_T^{-2} + Z_{pT}^T, \tag{6.9}$$

where

$$Z_{pT}^T = \frac{\left(\left(\frac{\Gamma_\mathrm{g}}{m_\mathrm{g} c_\mathrm{g}^2} + \frac{\Gamma_\ell}{m_\ell c_\ell^2} \right) \left(\frac{\Gamma_\mathrm{g}}{\rho_\mathrm{g} c_\mathrm{g}^2} - \frac{\Gamma_\ell}{\rho_\ell c_\ell^2} \right) m_\mathrm{g} c_\mathrm{g}^2 m_\ell c_\ell^2 - \alpha_\ell \alpha_\mathrm{g} \left(\frac{1}{C_{p,\mathrm{g}} T} + \frac{1}{C_{p,\ell} T} \right) (\rho_\mathrm{g} c_\mathrm{g}^2 - \rho_\ell c_\ell^2) \right)^2 \rho}{m_\ell c_\ell^2 m_\mathrm{g} c_\mathrm{g}^2 \left(\frac{1}{C_{p,\mathrm{g}} T} + \frac{1}{C_{p,\ell} T} \right) \left(\left(\frac{\Gamma_\mathrm{g}}{m_\mathrm{g} c_\mathrm{g}^2} + \frac{\Gamma_\ell}{m_\ell c_\ell^2} \right)^2 m_\ell c_\ell^2 m_\mathrm{g} c_\mathrm{g}^2 + \left(\frac{1}{C_{p,\mathrm{g}} T} + \frac{1}{C_{p,\ell} T} \right) \rho \tilde{a}_0^2 \right)}. \tag{6.10}$$

Proposition 3. *The mechanical-thermal equilibrium model satisfies the subcharacteristic condition with respect to the thermal equilibrium model of Section 4, subject only to the physically fundamental conditions*

$$\rho_k > 0,$$
$$c_{p,k} > 0,$$
$$T > 0.$$

Proof. By Eqs. (4.20) and (6.5), we see that the interlacing condition of Definition 1 reduces to the requirement that

$$\tilde{a}_T \geq \tilde{a}_{pT}, \tag{6.11}$$

which follows from Eqs. (6.9)–(6.10) and the given conditions for ρ_k, $c_{p,k}$ and T. □

7. Pressure-chemical relaxation. In this section, we investigate the model that arises when we impose volume and mass transfer equilibrium. In other words, we let the pressure and chemical potential relaxation parameters $\mathcal{J}, \mathcal{K}$ go to infinity. This corresponds to taking the limit

$$\mathcal{J} \to \infty \tag{7.1}$$

in the chemical equilibrium model (5.9)–(5.13), or equivalently the limit

$$\mathcal{K} \to \infty \tag{7.2}$$

in the mechanical equilibrium model of Section 3, which we expect to correspond to the assumptions

$$p_\mathrm{g} = p_\ell = p^* = p, \tag{7.3}$$
$$\mu_\mathrm{g} = \mu_\ell = \mu^* = \mu, \tag{7.4}$$

i.e. equal pressures and chemical potentials. This model was first introduced in this form by Flåtten and Lund [10].

7.1. The mechanical-chemical equilibrium model. The mechanical-chemical equilibrium model can be formulated as follows:

- Mass conservation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0, \tag{7.5}$$

- Momentum conservation:

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho v^2 + \alpha_{\mathrm{g}} p_{\mathrm{g}} + \alpha_\ell p_\ell)}{\partial x} = 0, \tag{7.6}$$

- Energy equations:

$$\begin{aligned}
&\frac{\partial E_{\mathrm{g}}}{\partial t} + \frac{\partial}{\partial x}(vE_{\mathrm{g}}) + \frac{m_{\mathrm{g}}}{\rho} v \frac{\partial p}{\partial x} + p\alpha_{\mathrm{g}} \frac{\rho \tilde{a}_p^2}{\rho_{\mathrm{g}} c_{\mathrm{g}}^2} \frac{\partial v}{\partial x} \\
&- \frac{\rho \tilde{a}_{p\mu}^2}{P_{\mathrm{g}}} \frac{m_{\mathrm{g}} s_{\mathrm{g}} T_{\mathrm{g}} C_{p,\ell}}{s_{\mathrm{g}}^2 T_{\mathrm{g}} C_{p,\ell} + s_\ell^2 T_\ell C_{p,\mathrm{g}}} \left(\mu + \frac{1}{2} v^2 + p \frac{\alpha_{\mathrm{g}}(\Gamma_\ell s_\ell T_\ell - c_\ell^2) + \alpha_\ell(\Gamma_{\mathrm{g}} s_{\mathrm{g}} T_{\mathrm{g}} - c_{\mathrm{g}}^2)}{\rho_{\mathrm{g}} \alpha_\ell c_{\mathrm{g}}^2 + \rho_\ell \alpha_{\mathrm{g}} c_\ell^2} \right) \frac{\partial v}{\partial x} \\
&= \left(1 - p \frac{\alpha_{\mathrm{g}} \Gamma_\ell + \alpha_\ell \Gamma_{\mathrm{g}}}{\rho_{\mathrm{g}} \alpha_\ell c_{\mathrm{g}}^2 + \rho_\ell \alpha_{\mathrm{g}} c_\ell^2} \right) \mathcal{H}(T_\ell - T_{\mathrm{g}}) \\
&+ \mathcal{H}(T_\ell - T_{\mathrm{g}}) \left(\mu + \frac{1}{2} v^2 + p \frac{\alpha_{\mathrm{g}}(\Gamma_\ell s_\ell T_\ell - c_\ell^2) + \alpha_\ell(\Gamma_{\mathrm{g}} s_{\mathrm{g}} T_{\mathrm{g}} - c_{\mathrm{g}}^2)}{\rho_{\mathrm{g}} \alpha_\ell c_{\mathrm{g}}^2 + \rho_\ell \alpha_{\mathrm{g}} c_\ell^2} \right) \\
&\cdot \left(\frac{s_{\mathrm{g}} C_{p,\ell} + s_\ell C_{p,\mathrm{g}}}{s_{\mathrm{g}}^2 T_{\mathrm{g}} C_{p,\ell} + s_\ell^2 T_\ell C_{p,\mathrm{g}}} \left(\frac{\tilde{a}_{p\mu}}{\tilde{a}_p} \right)^2 - \left(\frac{\Gamma_{\mathrm{g}}}{\rho_{\mathrm{g}} c_{\mathrm{g}}^2} - \frac{\Gamma_\ell}{\rho_\ell c_\ell^2} \right) \cdot \frac{\rho \tilde{a}_{p\mu}^2}{P_{\mathrm{g}}} \frac{\rho_{\mathrm{g}} \alpha_{\mathrm{g}} s_{\mathrm{g}} T_{\mathrm{g}} C_{p,\ell}}{s_{\mathrm{g}}^2 T_{\mathrm{g}} C_{p,\ell} + s_\ell^2 T_\ell C_{p,\mathrm{g}}} \right),
\end{aligned} \tag{7.7}$$

$$\begin{aligned}
&\frac{\partial E_\ell}{\partial t} + \frac{\partial}{\partial x}(vE_\ell) + \frac{m_\ell}{\rho} v \frac{\partial p}{\partial x} + p\alpha_\ell \frac{\rho \tilde{a}_p^2}{\rho_\ell c_\ell^2} \frac{\partial v}{\partial x} \\
&- \frac{\rho \tilde{a}_{p\mu}^2}{P_\ell} \frac{m_\ell s_\ell T_\ell C_{p,\mathrm{g}}}{s_{\mathrm{g}}^2 T_{\mathrm{g}} C_{p,\ell} + s_\ell^2 T_\ell C_{p,\mathrm{g}}} \left(\mu + \frac{1}{2} v^2 + p \frac{\alpha_{\mathrm{g}}(\Gamma_\ell s_\ell T_\ell - c_\ell^2) + \alpha_\ell(\Gamma_{\mathrm{g}} s_{\mathrm{g}} T_{\mathrm{g}} - c_{\mathrm{g}}^2)}{\rho_{\mathrm{g}} \alpha_\ell c_{\mathrm{g}}^2 + \rho_\ell \alpha_{\mathrm{g}} c_\ell^2} \right) \frac{\partial v}{\partial x} \\
&= \left(1 - p \frac{\alpha_{\mathrm{g}} \Gamma_\ell + \alpha_\ell \Gamma_{\mathrm{g}}}{\rho_{\mathrm{g}} \alpha_\ell c_{\mathrm{g}}^2 + \rho_\ell \alpha_{\mathrm{g}} c_\ell^2} \right) \mathcal{H}(T_{\mathrm{g}} - T_\ell) \\
&+ \mathcal{H}(T_{\mathrm{g}} - T_\ell) \left(\mu + \frac{1}{2} v^2 + p \frac{\alpha_{\mathrm{g}}(\Gamma_\ell s_\ell T_\ell - c_\ell^2) + \alpha_\ell(\Gamma_{\mathrm{g}} s_{\mathrm{g}} T_{\mathrm{g}} - c_{\mathrm{g}}^2)}{\rho_{\mathrm{g}} \alpha_\ell c_{\mathrm{g}}^2 + \rho_\ell \alpha_{\mathrm{g}} c_\ell^2} \right) \\
&\cdot \left(\frac{s_{\mathrm{g}} C_{p,\ell} + s_\ell C_{p,\mathrm{g}}}{s_{\mathrm{g}}^2 T_{\mathrm{g}} C_{p,\ell} + s_\ell^2 T_\ell C_{p,\mathrm{g}}} \left(\frac{\tilde{a}_{p\mu}}{\tilde{a}_p} \right)^2 + \left(\frac{\Gamma_{\mathrm{g}}}{\rho_{\mathrm{g}} c_{\mathrm{g}}^2} - \frac{\Gamma_\ell}{\rho_\ell c_\ell^2} \right) \cdot \frac{\rho \tilde{a}_{p\mu}^2}{P_\ell} \frac{\rho_\ell \alpha_\ell s_\ell T_\ell C_{p,\mathrm{g}}}{s_{\mathrm{g}}^2 T_{\mathrm{g}} C_{p,\ell} + s_\ell^2 T_\ell C_{p,\mathrm{g}}} \right).
\end{aligned} \tag{7.8}$$

As presented in Ref. [10], the energy equations (7.7)–(7.8) unfortunately contained a sign error, which has been corrected here. We have also introduced

$$P_{\mathrm{g}} \equiv \left(\frac{\partial p}{\partial s_{\mathrm{g}}} \right)_{s_\ell} = \frac{s_{\mathrm{g}} T_{\mathrm{g}}}{c_{p,\mathrm{g}}} \left(\frac{\xi_{\mathrm{g}}^2}{\rho_{\mathrm{g}} c_{\mathrm{g}}^2} - \frac{\xi_\ell^2}{\rho_\ell c_\ell^2} \right)^{-1}, \tag{7.9}$$

$$P_\ell \equiv \left(\frac{\partial p}{\partial s_\ell} \right)_{s_{\mathrm{g}}} = \frac{s_\ell T_\ell}{c_{p,\ell}} \left(\frac{\xi_\ell^2}{\rho_\ell c_\ell^2} - \frac{\xi_{\mathrm{g}}^2}{\rho_{\mathrm{g}} c_{\mathrm{g}}^2} \right)^{-1}, \tag{7.10}$$

and

$$\tilde{a}_{p\mu}^{-2} = \tilde{a}_p^{-2} + \frac{\rho C_{p,\mathrm{g}} C_{p,\ell}}{\rho_{\mathrm{g}}^2 \rho_\ell^2 (C_{p,\ell} s_{\mathrm{g}}^2 T_{\mathrm{g}} + C_{p,\mathrm{g}} s_\ell^2 T_\ell)} \left(\rho_{\mathrm{g}} - \rho_\ell + \rho_{\mathrm{g}} \rho_\ell \left(s_{\mathrm{g}} \frac{T_{\mathrm{g}} \Gamma_{\mathrm{g}}}{\rho_{\mathrm{g}} c_{\mathrm{g}}^2} - s_\ell \frac{T_\ell \Gamma_\ell}{\rho_\ell c_\ell^2} \right) \right)^2. \tag{7.11}$$

7.2. Wave velocities. The wave velocities of the mechanical-chemical equilibrium model (7.5)–(7.8) was analysed by Flåtten and Lund [10] in the non-stiff limit $\mathcal{H} \to 0$. The eigenvalues were found to be

$$\boldsymbol{\lambda}_{p\mu} = \{v - \tilde{a}_{p\mu}, v, v, v + \tilde{a}_{p\mu}\}, \tag{7.12}$$

where $\tilde{a}_{p\mu}$ is the mixture speed of sound, given by Eq. (7.11).

7.2.1. The subcharacteristic condition with respect to the p-model. From Eq. (7.11), we immediately see that the mixture speed of sound can be written as a sum of squares,

$$\tilde{a}_{p\mu}^{-2} = \tilde{a}_p^{-2} + Z_{p\mu}^p, \tag{7.13}$$

where

$$Z_{p\mu}^p = \frac{\rho C_{p,\mathrm{g}} C_{p,\ell}}{\rho_\mathrm{g}^2 \rho_\ell^2 (C_{p,\ell} s_\mathrm{g}^2 T_\mathrm{g} + C_{p,\mathrm{g}} s_\ell^2 T_\ell)} \left(\rho_\mathrm{g} - \rho_\ell + \rho_\mathrm{g}\rho_\ell \left(s_\mathrm{g} \frac{T_\mathrm{g}\Gamma_\mathrm{g}}{\rho_\mathrm{g} c_\mathrm{g}^2} - s_\ell \frac{T_\ell \Gamma_\ell}{\rho_\ell c_\ell^2} \right) \right)^2. \tag{7.14}$$

From this and Eqs. (3.2) and (7.12), we see that the subcharacteristic condition is satisfied, given only the physically fundamental conditions $\rho_k > 0$, $c_{p,k} > 0$, $T_k > 0$ [10].

7.2.2. The subcharacteristic condition with respect to the μ-model. Using the expressions for the mixture speed of sound in the chemical equilibrium model (5.15) and the present mechanical-chemical equilibrium model (7.11), it may be shown that the latter can be written as

$$\tilde{a}_{p\mu}^{-2} = \tilde{a}_\mu^{-2} + Z_{p\mu}^\mu, \tag{7.15}$$

where

$$Z_{p\mu}^\mu = \rho \frac{m_\mathrm{g} c_\mathrm{g}^2 m_\ell c_\ell^2 \left(\left(\frac{1}{\rho_\mathrm{g} c_\mathrm{g}^2} - \frac{1}{\rho_\ell c_\ell^2} \right) \left(\frac{s_\mathrm{g}^2 T_\mathrm{g}}{C_{p,\mathrm{g}}} + \frac{s_\ell^2 T_\ell}{C_{p,\ell}} \right) + \left(\frac{\xi_\mathrm{g}^2}{m_\mathrm{g} c_\mathrm{g}^2} + \frac{\xi_\ell^2}{m_\ell c_\ell^2} \right) \left(\frac{\xi_\mathrm{g}^2}{\rho_\mathrm{g} c_\mathrm{g}^2} - \frac{\xi_\ell^2}{\rho_\ell c_\ell^2} \right) \right)^2}{\left(\frac{s_\mathrm{g}^2 T_\mathrm{g}}{C_{p,\mathrm{g}}} + \frac{s_\ell^2 T_\ell}{C_{p,\ell}} \right) \left(\left(\frac{\xi_\mathrm{g}^2}{m_\mathrm{g} c_\mathrm{g}^2} + \frac{\xi_\ell^2}{m_\ell c_\ell^2} \right)^2 c_\ell^2 c_\mathrm{g}^2 m_\mathrm{g} m_\ell + \left(\frac{s_\ell^2 T_\ell}{C_{p,\ell}} + \frac{s_\mathrm{g}^2 T_\mathrm{g}}{C_{p,\mathrm{g}}} \right) \rho \tilde{a}_0^2 \right)}. \tag{7.16}$$

PROPOSITION 4. *The mechanical-chemical equilibrium model given by Eqs.* (7.5)–(7.8) *satisfies the subcharacteristic condition with respect to the chemical equilibrium model of Section 5, subject only to the physically fundamental conditions*

$$\rho_k > 0,$$
$$c_{p,k} > 0,$$
$$T_k > 0.$$

Proof. By Eqs. (5.19) and (7.12), we see that the interlacing condition of Definition 1 reduces to the requirement that

$$\tilde{a}_\mu \geq \tilde{a}_{p\mu}, \tag{7.17}$$

which follows from Eqs. (7.15)–(7.16) and the given conditions for ρ_k, $c_{p,k}$ and T_k. $\square$

8. Temperature-chemical relaxation. In this section, we investigate the model that results when we assume heat and mass transfer equilibrium, in other words that the relaxation parameters $\mathcal{H}, \mathcal{K}$ go to infinity. This is equivalent to taking the limit

$$\mathcal{H} \to \infty \tag{8.1}$$

in the thermal equilibrium model of Section 4, or equivalently the limit

$$\mathcal{K} \to \infty \tag{8.2}$$

in the chemical equilibrium model of Section 5. We expect this to be equivalent to the assumptions

$$T_\mathrm{g} = T_\ell = T, \tag{8.3}$$

$$\mu_\mathrm{g} = \mu_\ell = \mu^* = \mu, \tag{8.4}$$

i.e. thermal and chemical equilibrium. The model equations and wave velocities for this model have not been found elsewhere, and will thus be derived here.

8.1. Entropy equations. To derive the entropy equations of the thermal-chemical equilibrium model, we start by adding the balance formulations of the entropy equations (2.13)–(2.14) to eliminate the heat transfer term, which, after expanding and rewriting derivatives, yields

$$T\left(m_{\mathrm{g}}\mathrm{D}_t s_{\mathrm{g}} + m_\ell \mathrm{D}_t s_\ell + s_{\mathrm{g}}\alpha_{\mathrm{g}}\mathrm{D}_t\rho_{\mathrm{g}} + s_\ell\alpha_\ell\mathrm{D}_t\rho_\ell + (m_\ell s_\ell + m_{\mathrm{g}}s_{\mathrm{g}})\frac{\partial v}{\partial x}\right) = (p_{\mathrm{g}} - p_\ell - T(s_{\mathrm{g}}\rho_{\mathrm{g}} - s_\ell\rho_\ell))\mathcal{J}(p_{\mathrm{g}} - p_\ell), \tag{8.5}$$

where we also have let $\mathcal{K}$ go to infinity, hence eliminating the mass transfer term.

To eliminate the material derivative $\mathrm{D}_t\rho_k$, we need to establish certain differentials. Since the chemical potentials and temperatures are equal, so are their differentials, which gives us

$$\mathrm{d}\mu = \frac{1}{\rho_\ell}\mathrm{d}p_\ell - s_\ell\mathrm{d}T = \frac{1}{\rho_{\mathrm{g}}}\mathrm{d}p_{\mathrm{g}} - s_{\mathrm{g}}\mathrm{d}T, \tag{8.6}$$

$$\mathrm{d}T = \frac{\Gamma_{\mathrm{g}}T}{\rho_{\mathrm{g}}c_{\mathrm{g}}^2}\mathrm{d}p_{\mathrm{g}} + \frac{T}{c_{p,\mathrm{g}}}\mathrm{d}s_{\mathrm{g}} = \frac{\Gamma_\ell T}{\rho_\ell c_\ell^2}\mathrm{d}p_\ell + \frac{T}{c_{p,\ell}}\mathrm{d}s_\ell, \tag{8.7}$$

$$\mathrm{d}p_k = c_k^2\mathrm{d}\rho_k + \rho_k\Gamma_k T_k\mathrm{d}s_k. \tag{8.8}$$

Solving these three equations for $\mathrm{d}\rho_k$ as functions of $\mathrm{d}s_{\mathrm{g}}$ and $\mathrm{d}s_\ell$ yields

$$\mathrm{d}\rho_{\mathrm{g}} = \rho_{\mathrm{g}}\frac{\left(\frac{1}{c_{p,\mathrm{g}}}\left(-\frac{\Gamma_\ell}{c_\ell^2}\Delta h - 1\right) - \Gamma_{\mathrm{g}}T\left(\frac{\Gamma_{\mathrm{g}}}{c_{\mathrm{g}}^2} - \frac{\Gamma_\ell}{c_\ell^2} + \frac{\Gamma_\ell\Gamma_{\mathrm{g}}}{c_{\mathrm{g}}^2c_\ell^2}\Delta h\right)\right)\mathrm{d}s_{\mathrm{g}} + \frac{1}{c_{p,\ell}}\mathrm{d}s_\ell}{c_{\mathrm{g}}^2\left(\frac{\Gamma_{\mathrm{g}}}{c_{\mathrm{g}}^2} - \frac{\Gamma_\ell}{c_\ell^2} + \frac{\Gamma_\ell\Gamma_{\mathrm{g}}}{c_{\mathrm{g}}^2c_\ell^2}\Delta h\right)}, \tag{8.9}$$

$$\mathrm{d}\rho_\ell = \rho_\ell\frac{\left(\frac{1}{c_{p,\ell}}\left(\frac{\Gamma_{\mathrm{g}}}{c_{\mathrm{g}}^2}\Delta h - 1\right) - \Gamma_\ell T\left(\frac{\Gamma_\ell}{c_\ell^2} - \frac{\Gamma_{\mathrm{g}}}{c_{\mathrm{g}}^2} - \frac{\Gamma_{\mathrm{g}}\Gamma_\ell}{c_\ell^2c_{\mathrm{g}}^2}\Delta h\right)\right)\mathrm{d}s_\ell + \frac{1}{c_{p,\mathrm{g}}}\mathrm{d}s_{\mathrm{g}}}{c_\ell^2\left(\frac{\Gamma_\ell}{c_\ell^2} - \frac{\Gamma_{\mathrm{g}}}{c_{\mathrm{g}}^2} - \frac{\Gamma_{\mathrm{g}}\Gamma_\ell}{c_\ell^2c_{\mathrm{g}}^2}\Delta h\right)}, \tag{8.10}$$

where $\Delta h \equiv h_{\mathrm{g}} - h_\ell$. We will also have use for the differential of the mixture density,

$$\mathrm{d}\rho = \alpha_{\mathrm{g}}\mathrm{d}\rho_{\mathrm{g}} + \alpha_\ell\mathrm{d}\rho_\ell + (\rho_{\mathrm{g}} - \rho_\ell)\mathrm{d}\alpha_{\mathrm{g}}. \tag{8.11}$$

We may now express Eqs. (8.9)–(8.11) using the material derivative, which together with Eq. (8.5) allows us to solve for the entropy equations, which turn out to be slightly complex,

$$\begin{aligned}
\mathrm{D}_t s_{\mathrm{g}} = C_{p,\mathrm{g}}\Bigg[&\left(\left(\Delta h\frac{\Gamma_{\mathrm{g}}}{c_{\mathrm{g}}^2}\frac{\Gamma_\ell}{c_\ell^2} + \frac{\Gamma_{\mathrm{g}}}{c_{\mathrm{g}}^2} - \frac{\Gamma_\ell}{c_\ell^2}\right)\left(\rho + m_{\mathrm{g}}\Delta h\frac{\Gamma_\ell}{c_\ell^2}\right)C_{p,\ell}Tc_{\mathrm{g}}^2c_\ell^2 + \Delta h m_\ell m_{\mathrm{g}}(c_\ell^2 - c_{\mathrm{g}}^2 + \Delta h\Gamma_{\mathrm{g}})\right)\frac{\partial v}{\partial x} \\
&+ \left(\left(\Delta h\frac{\Gamma_{\mathrm{g}}}{c_{\mathrm{g}}^2}\frac{\Gamma_\ell}{c_\ell^2} + \frac{\Gamma_{\mathrm{g}}}{c_{\mathrm{g}}^2} - \frac{\Gamma_\ell}{c_\ell^2}\right)\left((\Delta h\rho_{\mathrm{g}} - \Delta p)\frac{\Gamma_\ell}{c_\ell^2} + \rho_{\mathrm{g}} - \rho_\ell\right)C_{p,\ell}Tc_\ell^2c_{\mathrm{g}}^2\right. \\
&\left. + \left(\Delta h\frac{\Gamma_{\mathrm{g}}}{c_{\mathrm{g}}^2} - 1\right)(\Delta h\rho_{\mathrm{g}} - \Delta p)m_\ell c_{\mathrm{g}}^2 - (\Delta h\rho_\ell - \Delta p)c_\ell^2 m_{\mathrm{g}}\right)\mathcal{J}(p_{\mathrm{g}} - p_\ell)\Bigg] \\
\times\Bigg[&\left(\left(\Delta h\frac{\Gamma_{\mathrm{g}}}{c_{\mathrm{g}}^2}\frac{\Gamma_\ell}{c_\ell^2} + \frac{\Gamma_{\mathrm{g}}}{c_{\mathrm{g}}^2} - \frac{\Gamma_\ell}{c_\ell^2}\right)^2 c_\ell^2c_{\mathrm{g}}^2C_{p,\mathrm{g}}C_{p,\ell}T^2 + c_{\mathrm{g}}^2m_\ell C_{p,\ell}T + c_\ell^2m_{\mathrm{g}}C_{p,\mathrm{g}}T + \Delta h^2m_{\mathrm{g}}m_\ell\right. \\
&\left. + \left(\Delta h\frac{\Gamma_\ell}{c_\ell^2} + 1\right)^2 c_\ell^2m_{\mathrm{g}}C_{p,\ell}T + \left(\Delta h\frac{\Gamma_{\mathrm{g}}}{c_{\mathrm{g}}^2} - 1\right)^2 m_\ell C_{p,\mathrm{g}}Tc_{\mathrm{g}}^2\right)m_{\mathrm{g}}\Bigg]^{-1},
\end{aligned} \tag{8.12}$$

(8.13)

$$\begin{aligned}\mathrm{D}_t s_\ell = C_{p,\ell}\Bigg[&\Bigg(\left(-\Delta h\frac{\Gamma_\ell}{c_\ell^2}\frac{\Gamma_\mathrm{g}}{c_\mathrm{g}^2}+\frac{\Gamma_\ell}{c_\ell^2}-\frac{\Gamma_\mathrm{g}}{c_\mathrm{g}^2}\right)\left(\rho-m_\ell\Delta h\frac{\Gamma_\mathrm{g}}{c_\mathrm{g}^2}\right)C_{p,\mathrm{g}}Tc_\ell^2c_\mathrm{g}^2-\Delta h m_\mathrm{g}m_\ell\left(c_\mathrm{g}^2-c_\ell^2-\Delta h\Gamma_\ell\right)\Bigg)\frac{\partial v}{\partial x}\\
&+\Bigg(\left(\Delta h\frac{\Gamma_\ell}{c_\ell^2}\frac{\Gamma_\mathrm{g}}{c_\mathrm{g}^2}-\frac{\Gamma_\ell}{c_\ell^2}+\frac{\Gamma_\mathrm{g}}{c_\mathrm{g}^2}\right)\left((\Delta h\rho_\ell-\Delta p)\frac{\Gamma_\mathrm{g}}{c_\mathrm{g}^2}-\rho_\ell+\rho_\mathrm{g}\right)C_{p,\mathrm{g}}Tc_\mathrm{g}^2c_\ell^2\\
&+\left(\Delta h\frac{\Gamma_\ell}{c_\ell^2}+1\right)(\Delta h\rho_\ell-\Delta p)\,m_\mathrm{g}c_\ell^2+(\Delta h\rho_\mathrm{g}-\Delta p)\,c_\mathrm{g}^2m_\ell\Bigg)\mathcal{J}(p_\ell-p_\mathrm{g})\Bigg]\\
\times\Bigg[&\Bigg(\left(\Delta h\frac{\Gamma_\ell}{c_\ell^2}\frac{\Gamma_\mathrm{g}}{c_\mathrm{g}^2}-\frac{\Gamma_\ell}{c_\ell^2}+\frac{\Gamma_\mathrm{g}}{c_\mathrm{g}^2}\right)^2c_\mathrm{g}^2c_\ell^2C_{p,\ell}C_{p,\mathrm{g}}T^2+c_\ell^2m_\mathrm{g}C_{p,\mathrm{g}}T+c_\mathrm{g}^2m_\ell C_{p,\ell}T+\Delta h^2m_\ell m_\mathrm{g}\\
&+\left(\Delta h\frac{\Gamma_\mathrm{g}}{c_\mathrm{g}^2}-1\right)^2c_\mathrm{g}^2m_\ell C_{p,\mathrm{g}}T+\left(\Delta h\frac{\Gamma_\ell}{c_\ell^2}+1\right)^2m_\mathrm{g}C_{p,\ell}Tc_\ell^2\Bigg)m_\ell\Bigg]^{-1},\end{aligned}$$

where we have used Eqs. (2.3) and (2.5) to replace $\mathrm{D}_t\rho$ and $\mathrm{D}_t\alpha_\mathrm{g}$, and introduced $\Delta p \equiv p_\mathrm{g}-p_\ell$.

8.2. The thermal-chemical equilibrium model. The thermal-chemical equilibrium model can be formulated as follows:

- Mass conservation:

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho v)}{\partial x}=0, \tag{8.14}$$

- Momentum conservation:

$$\frac{\partial(\rho v)}{\partial t}+\frac{\partial(\rho v^2+\alpha_\mathrm{g}p_\mathrm{g}+\alpha_\ell p_\ell)}{\partial x}=0, \tag{8.15}$$

- Volume advection:

$$\mathrm{D}_t\alpha_\mathrm{g}=\mathcal{J}(p_\mathrm{g}-p_\ell), \tag{8.16}$$

- Energy conservation:

$$\frac{\partial E}{\partial t}+\frac{\partial(v(E+p))}{\partial x}=0. \tag{8.17}$$

An alternative formulation may be obtained by using the more obscure entropy equations (8.12)–(8.13) instead of the volume fraction (8.16) and energy equations (8.17).

8.3. Wave velocities. We now wish to calculate the wave velocities, and hence the mixture speed of sound, of the thermal-chemical equilibrium model (8.14)–(8.17) in the non-stiff limit where $\mathcal{J}\to 0$. To this end, we find it useful to derive an evolution equation for the effective pressure p_eff.

We express Eqs. (8.8)–(8.10) using the material derivative, which together with Eqs. (8.12)–(8.13), (8.16) and (4.14) yields

$$\mathrm{D}_t p_\mathrm{eff}=-\rho\tilde{a}_{T\mu}^2\frac{\partial v}{\partial x}, \tag{8.18}$$

where

$$\begin{aligned}\tilde{a}_{T\mu}^2=&\left(\frac{\rho}{C_{p,\mathrm{g}}T}\left(1+\Delta h\frac{\Gamma_\ell}{c_\ell^2}\frac{m_\mathrm{g}}{\rho}\right)^2+\frac{\rho}{C_{p,\ell}T}\left(1-\Delta h\frac{\Gamma_\mathrm{g}}{c_\mathrm{g}^2}\frac{m_\ell}{\rho}\right)^2+\frac{\Delta h^2m_\mathrm{g}m_\ell\tilde{a}_0^2}{C_{p,\mathrm{g}}C_{p,\ell}T^2c_\mathrm{g}^2c_\ell^2}\right)\\
&\times\Bigg[\frac{m_\ell}{c_\ell^2C_{p,\mathrm{g}}T}+\frac{m_\mathrm{g}}{c_\mathrm{g}^2C_{p,\ell}T}+\left(\Delta h\frac{\Gamma_\mathrm{g}}{c_\mathrm{g}^2}-1\right)^2\frac{m_\ell}{c_\ell^2C_{p,\ell}T}+\left(\Delta h\frac{\Gamma_\ell}{c_\ell^2}+1\right)^2\frac{m_\mathrm{g}}{c_\mathrm{g}^2C_{p,\mathrm{g}}T}\\
&\qquad+\frac{\Delta h^2m_\ell m_\mathrm{g}}{C_{p,\mathrm{g}}C_{p,\ell}T^2c_\mathrm{g}^2c_\ell^2}+\left(\Delta h\frac{\Gamma_\mathrm{g}}{c_\mathrm{g}^2}\frac{\Gamma_\ell}{c_\ell^2}+\frac{\Gamma_\mathrm{g}}{c_\mathrm{g}^2}-\frac{\Gamma_\ell}{c_\ell^2}\right)^2\Bigg]^{-1}.\end{aligned} \tag{8.19}$$

From Eq. (8.16), we know that α_{g} is a characteristic variable with the corresponding eigenvalue v. The remaining equations (8.14), (8.15) and (8.18) may then be expressed as a quasi-linear equation system in the variables $\boldsymbol{u} = [\rho, \rho v, p_{\mathrm{eff}}]$,

$$\boldsymbol{u}_t + \begin{bmatrix} 0 & 1 & 0 \\ -v^2 & 2v & 1 \\ -v\tilde{a}_{T\mu}^2 & \tilde{a}_{T\mu}^2 & v \end{bmatrix} \boldsymbol{u}_x = 0. \tag{8.20}$$

The eigenvalues of this system are $\{\tilde{a}_{T\mu}, v, v + \tilde{a}_{T\mu}\}$, thus the eigenvalues of the full model may be summarised as

$$\boldsymbol{\lambda}_{T\mu} = \{v - \tilde{a}_{T\mu}, v, v, v + \tilde{a}_{T\mu}\}. \tag{8.21}$$

8.3.1. The subcharacteristic condition with respect to T-model. Using Eqs. (4.17) and (8.19), it may be shown that the mixture speed of sound of the present model may be written as

$$\tilde{a}_{T\mu}^{-2} = \tilde{a}_T^{-2} + Z_{T\mu}^T, \tag{8.22}$$

where

$$\begin{aligned} Z_{T\mu}^T = {} & \Bigg(\Delta h \left(\frac{\Gamma_{\mathrm{g}}}{C_{p,\ell}T} + \frac{\Gamma_\ell}{C_{p,\mathrm{g}}T} \right) + (c_\ell^2 - c_{\mathrm{g}}^2) \left(\frac{1}{C_{p,\ell}T} + \frac{1}{C_{p,\mathrm{g}}T} \right) \\ & + \left(\frac{\Gamma_\ell}{m_\ell c_\ell^2} + \frac{\Gamma_{\mathrm{g}}}{m_{\mathrm{g}} c_{\mathrm{g}}^2} \right) \left(\Delta h \frac{\Gamma_{\mathrm{g}}\Gamma_\ell}{c_{\mathrm{g}}^2 c_\ell^2} + \frac{\Gamma_{\mathrm{g}}}{c_{\mathrm{g}}^2} - \frac{\Gamma_\ell}{c_\ell^2} \right) c_\ell^2 c_{\mathrm{g}}^2 \Bigg)^2 m_{\mathrm{g}} m_\ell \\ & \times \Bigg[\Bigg(\Bigg(\frac{\rho}{C_{p,\ell}T} \left(1 - \Delta h \frac{m_\ell}{\rho} \frac{\Gamma_{\mathrm{g}}}{c_{\mathrm{g}}^2} \right)^2 + \frac{\rho}{C_{p,\mathrm{g}}T} \left(1 + \Delta h \frac{m_{\mathrm{g}}}{\rho} \frac{\Gamma_\ell}{c_\ell^2} \right)^2 \Bigg) c_\ell^2 c_{\mathrm{g}}^2 + \Delta h^2 \frac{m_{\mathrm{g}}}{C_{p,\mathrm{g}}T} \frac{m_\ell}{C_{p,\ell}T} \tilde{a}_0^2 \Bigg) \\ & \Bigg(\left(\frac{1}{C_{p,\mathrm{g}}T} + \frac{1}{C_{p,\ell}T} \right) \rho \tilde{a}_0^2 + \left(\frac{\Gamma_\ell}{m_\ell c_\ell^2} + \frac{\Gamma_{\mathrm{g}}}{m_{\mathrm{g}} c_{\mathrm{g}}^2} \right)^2 m_{\mathrm{g}} m_\ell c_\ell^2 c_{\mathrm{g}}^2 \Bigg) \Bigg]^{-1}. \end{aligned} \tag{8.23}$$

PROPOSITION 5. *The thermal-chemical equilibrium model given by Eqs.* (8.14)–(8.17) *satisfies the subcharacteristic condition with respect to the thermal equilibrium model of Section 4, subject only to the physically fundamental conditions*

$$\begin{aligned} \rho_k &> 0, \\ c_{p,k} &> 0, \\ T &> 0. \end{aligned}$$

Proof. By Eqs. (4.20) and (8.21), we see that the interlacing condition from Definition 1 reduces to the requirement that

$$\tilde{a}_T \geq \tilde{a}_{T\mu}, \tag{8.24}$$

which follows from Eqs. (8.22)–(8.23) and the given conditions for ρ_k, $c_{p,k}$ and T. □

8.3.2. The subcharacteristic condition with respect to the μ-model. From Eqs. (5.15) and (8.19), we find that the mixture speed of sound in the present model may be written as

$$\tilde{a}_{T\mu}^{-2} = \tilde{a}_\mu^{-2} + Z_{T\mu}^\mu, \tag{8.25}$$

where

$$Z^{\mu}_{T\mu} = \Bigg(-(\Delta h \Gamma_{\mathrm{g}} - c_{\mathrm{g}}^2 + c_\ell^2)\frac{s_\ell}{C_{p,\ell}} - (\Delta h \Gamma_\ell + c_\ell^2 - c_{\mathrm{g}}^2)\frac{s_{\mathrm{g}}}{C_{p,\mathrm{g}}}$$
$$+ \left(\frac{\rho}{m_{\mathrm{g}} m_\ell} - \frac{\Gamma_\ell s_\ell T}{m_\ell c_\ell^2} - \frac{\Gamma_{\mathrm{g}} s_{\mathrm{g}} T}{m_{\mathrm{g}} c_{\mathrm{g}}^2}\right)\left(\Delta h \frac{\Gamma_{\mathrm{g}}}{c_{\mathrm{g}}^2}\frac{\Gamma_\ell}{c_\ell^2} + \frac{\Gamma_{\mathrm{g}}}{c_{\mathrm{g}}^2} - \frac{\Gamma_\ell}{c_\ell^2}\right) c_\ell^2 c_{\mathrm{g}}^2\Bigg)^2$$
$$\times \Bigg[\left(\frac{\Delta h^2 \tilde{a}_0^2}{C_{p,\mathrm{g}} C_{p,\ell} T^2} + \left(\frac{1}{C_{p,\mathrm{g}} T}\left(\frac{\rho}{m_{\mathrm{g}} m_\ell} + \frac{\Gamma_\ell \Delta h}{m_\ell c_\ell^2}\right)^2 + \frac{1}{C_{p,\ell} T}\left(\frac{\rho}{m_{\mathrm{g}} m_\ell} - \frac{\Gamma_{\mathrm{g}} \Delta h}{m_{\mathrm{g}} c_{\mathrm{g}}^2}\right)^2\right)\frac{m_\ell c_\ell^2 m_{\mathrm{g}} c_{\mathrm{g}}^2}{\rho}\right)$$
$$\left(\left(\frac{s_{\mathrm{g}}^2 T}{C_{p,\mathrm{g}}} + \frac{s_\ell^2 T}{C_{p,\ell}}\right)\rho \tilde{a}_0^2 + m_{\mathrm{g}} m_\ell \left(\frac{\rho}{m_{\mathrm{g}} m_\ell} - \frac{\Gamma_\ell s_\ell T}{m_\ell c_\ell^2} - \frac{\Gamma_{\mathrm{g}} s_{\mathrm{g}} T}{m_{\mathrm{g}} c_{\mathrm{g}}^2}\right)^2 c_\ell^2 c_{\mathrm{g}}^2\right)\Bigg]^{-1}. \tag{8.26}$$

PROPOSITION 6. *The thermal-chemical equilibrium model given by Eqs.* (8.14)–(8.17) *satisfies the subcharacteristic condition with respect to the chemical equilibrium model of Section 5, subject only to the physically fundamental conditions*

$$\rho_k > 0,$$
$$c_{p,k} > 0,$$
$$T > 0.$$

Proof. By Eqs. (5.19) and (8.21), we see that the interlacing condition of Definition 1 reduces to the requirement that

$$\tilde{a}_\mu \geq \tilde{a}_{T\mu}, \tag{8.27}$$

which follows from Eqs. (8.25)–(8.26) and the given conditions for ρ_k, $c_{p,k}$ and T. □

9. Full relaxation. In this section, we investigate the model that results when we let all the relaxation parameters $\mathcal{J}, \mathcal{H}, \mathcal{K}$ in the basic model of Section 2 go to infinity. We expect this to correspond to the assumptions

$$p_{\mathrm{g}} = p_\ell = p^* = p, \tag{9.1}$$
$$T_{\mathrm{g}} = T_\ell = T, \tag{9.2}$$
$$\mu_{\mathrm{g}} = \mu_\ell = \mu^* = \mu. \tag{9.3}$$

In other words, the two phases are in full equilibrium. This model is also referred to as the *homogeneous equilibrium model* [26], and has been used for two-phase flow simulations by a number of authors [8, 19].

9.1. The full equilibrium model. The full equilibrium model can be formulated through conservation equations for total mass, momentum and energy:

- Total mass conservation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial x} = 0, \tag{9.4}$$

- Momentum conservation:

$$\frac{\partial (\rho v)}{\partial t} + \frac{\partial (\rho v^2 + p)}{\partial x} = 0, \tag{9.5}$$

- Total energy conservation:

$$\frac{\partial E}{\partial t} + \frac{\partial (v(E+p))}{\partial x} = 0. \tag{9.6}$$

Here, the energy equation (9.6) is obtained simply by adding the energy equations (2.8)–(2.9) of the basic model.

9.2. Wave velocities. The wave velocities of the full equilibrium model have been analysed by e.g. Städtke [26], Saurel et al. [23] and Flåtten and Lund [10]. The eigenvalues are given by

$$\boldsymbol{\lambda}_{pT\mu} = \{v - \tilde{a}_{pT\mu}, v, v + \tilde{a}_{pT\mu}\}, \tag{9.7}$$

where the mixture speed of sound is given by [23]

$$\tilde{a}_{pT\mu}^{-2} = \tilde{a}_p^{-2} + \rho T \left[\frac{\alpha_{\mathrm{g}}\rho_{\mathrm{g}}}{c_{p,\mathrm{g}}} \left(\frac{\partial s_{\mathrm{g}}}{\partial p}\right)_{\mathrm{sat}}^2 + \frac{\alpha_\ell \rho_\ell}{c_{p,\ell}} \left(\frac{\partial s_\ell}{\partial p}\right)_{\mathrm{sat}}^2 \right], \tag{9.8}$$

where the notation $(\cdot)_{\mathrm{sat}}$ is used for differentiation along the boiling curve. The mixture speed of sound may also be expressed through the thermodynamic derivatives used earlier (Γ_k, c_k and $c_{p,k}$), by replacing the saturation derivative using

$$\left(\frac{\partial s_k}{\partial p}\right)_{\mathrm{sat}} = -\frac{\Gamma_k c_{p,k}}{\rho_k c_k^2} - \frac{c_{p,k}(\rho_{\mathrm{g}} - \rho_\ell)}{\rho_{\mathrm{g}}\rho_\ell(h_{\mathrm{g}} - h_\ell)}. \tag{9.9}$$

9.2.1. The subcharacteristic condition with respect to the pT-model. As shown by Flåtten and Lund [10], the subcharacteristic condition with respect to the mechanical-thermal equilibrium model of Section 6 is satisfied, given only $\rho_k > 0$, $c_{p,k} > 0$ and $T > 0$, which was shown by writing

$$\tilde{a}_{pT\mu}^{-2} = \tilde{a}_{pT}^{-2} + Z_{pT\mu}^{pT}, \tag{9.10}$$

where

$$Z_{pT\mu}^{pT} = \frac{\rho T}{C_{p,\mathrm{g}} + C_{p,\ell}} \left(\frac{\rho_{\mathrm{g}} - \rho_\ell}{\rho_{\mathrm{g}}\rho_\ell(h_{\mathrm{g}} - h_\ell)}(C_{p,\mathrm{g}} + C_{p,\ell}) + \frac{\Gamma_{\mathrm{g}} C_{p,\mathrm{g}}}{\rho_{\mathrm{g}} c_{\mathrm{g}}^2} + \frac{\Gamma_\ell C_{p,\ell}}{\rho_\ell c_\ell^2} \right)^2. \tag{9.11}$$

9.2.2. The subcharacteristic condition with respect to the $p\mu$-model. Also shown by Flåtten and Lund [10], the full equilibrium model fulfils the subcharacteristic condition with respect to the mechanical-chemical equilibrium model of Section 7, given only $\rho_k > 0$, $c_{p,k} > 0$ and $T > 0$, which may be shown by writing

$$\tilde{a}_{pT\mu}^{-2} = \tilde{a}_{p\mu}^{-2} + Z_{pT\mu}^{p\mu}, \tag{9.12}$$

where

$$\begin{aligned} Z_{pT\mu}^{p\mu} = \frac{\rho}{T(C_{p,\ell}s_{\mathrm{g}}^2 + C_{p,\mathrm{g}}s_\ell^2)} \Bigg(&\frac{(\rho_\ell - \rho_{\mathrm{g}})(C_{p,\mathrm{g}}s_\ell + C_{p,\ell}s_{\mathrm{g}})}{\rho_{\mathrm{g}}\rho_\ell(s_\ell - s_{\mathrm{g}})} \\ &+ T\frac{C_{p,\mathrm{g}}C_{p,\ell}s_{\mathrm{g}}s_\ell\left(\frac{\Gamma_\ell}{\rho_\ell c_\ell^2} + \frac{\Gamma_{\mathrm{g}}}{\rho_{\mathrm{g}}c_{\mathrm{g}}^2}\right) + \frac{\Gamma_{\mathrm{g}}}{\rho_{\mathrm{g}}c_{\mathrm{g}}^2}C_{p,\mathrm{g}}^2 s_\ell^2 + \frac{\Gamma_\ell}{\rho_\ell c_\ell^2}C_{p,\ell}^2 s_{\mathrm{g}}^2}{C_{p,\mathrm{g}}s_\ell + C_{p,\ell}s_{\mathrm{g}}} \Bigg)^2. \end{aligned} \tag{9.13}$$

9.2.3. The subcharacteristic condition with respect to the $T\mu$-model. By algebraic manipulations, one may show that the mixture speed of sound of the full equilibrium model is related to the one of the thermal-chemical equilibrium model as given by

$$\tilde{a}_{pT\mu}^{-2} = \tilde{a}_{T\mu}^{-2} + Z_{pT\mu}^{T\mu}, \tag{9.14}$$

where

$$\begin{aligned} Z_{pT\mu}^{T\mu} = \Bigg(\Bigg(&C_{p,\ell}T\left(\rho_{\mathrm{g}} - \rho_\ell + \Delta h\frac{\Gamma_\ell}{c_\ell^2}\rho_{\mathrm{g}}\right)\left(\rho + \Delta h\frac{\Gamma_\ell}{c_\ell^2}m_{\mathrm{g}}\right) \\ &- C_{p,\mathrm{g}}T\left(\rho_\ell - \rho_{\mathrm{g}} - \Delta h\frac{\Gamma_{\mathrm{g}}}{c_{\mathrm{g}}^2}\rho_\ell\right)\left(\rho - \Delta h\frac{\Gamma_{\mathrm{g}}}{c_{\mathrm{g}}^2}m_\ell\right) + \Delta h^2\frac{m_{\mathrm{g}}m_\ell}{c_{\mathrm{g}}^2c_\ell^2}(c_{\mathrm{g}}^2\rho_{\mathrm{g}} - c_\ell^2\rho_\ell)\Bigg)^2 \rho\Bigg) \\ &\cdot \left[\left(C_{p,\mathrm{g}}T\left(\rho - m_\ell\Delta h\frac{\Gamma_{\mathrm{g}}}{c_{\mathrm{g}}^2}\right)^2 + C_{p,\ell}T\left(\rho + m_{\mathrm{g}}\Delta h\frac{\Gamma_\ell}{c_\ell^2}\right)^2 + \frac{m_{\mathrm{g}}m_\ell}{c_{\mathrm{g}}^2c_\ell^2}\Delta h^2\rho\tilde{a}_0^2\right)\Delta h^2\rho_{\mathrm{g}}^2\rho_\ell^2\right]^{-1}. \end{aligned} \tag{9.15}$$

PROPOSITION 7. *The full equilibrium model given by Eqs.* (9.4)–(9.6) *satisfies the subcharacteristic condition with respect to the thermal-chemical equilibrium model of Section 8, given only the physically fundamental requirements*

$$\begin{aligned}\rho_k &> 0,\\ c_{p,k} &> 0,\\ T &> 0.\end{aligned}$$

Proof. From Eqs. (8.21) and (9.7), we find that the interlacing condition in Definition 1 translates to the requirement that

$$\tilde{a}_{p\mu} \geq \tilde{a}_{pT\mu}, \tag{9.16}$$

which follows from Eqs. (9.14)–(9.15) and the given conditions for ρ_k, $c_{p,k}$ and T. □

9.2.4. The discontinuity of the speed of sound. We have now considered eight different models with varying equilibrium assumptions, each with its own speed of sound. One would expect that the two-phase speed of sound reduces to the single-phase speed of sound in the limit where one phase disappears, which is indeed the case with seven of the models,

$$\lim_{\alpha_k\to 1}\tilde{a}_0 = \lim_{\alpha_k\to 1}\tilde{a}_p = \lim_{\alpha_k\to 1}\tilde{a}_T = \lim_{\alpha_k\to 1}\tilde{a}_\mu = \lim_{\alpha_k\to 1}\tilde{a}_{pT} = \lim_{\alpha_k\to 1}\tilde{a}_{p\mu} = \lim_{\alpha_k\to 1}\tilde{a}_{T\mu} = c_k. \tag{9.17}$$

However, for the final and present full equilibrium model, the single phase limit of the two-phase speed of sound turns out to be discontinuous,

$$\lim_{\alpha_{\mathrm{g}}\to 1}\tilde{a}_{pT\mu} = \left(\frac{1}{c_{\mathrm{g}}^2} + c_{p,\mathrm{g}}T\left(\frac{\rho_{\mathrm{g}}-\rho_\ell}{\rho_\ell(h_{\mathrm{g}}-h_\ell)} + \frac{\Gamma_{\mathrm{g}}}{c_{\mathrm{g}}^2}\right)^2\right)^{-\frac{1}{2}} \neq c_{\mathrm{g}}, \tag{9.18}$$

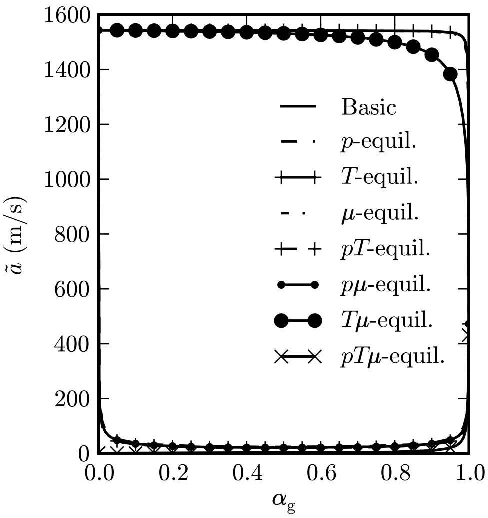
$$\lim_{\alpha_\ell\to 1}\tilde{a}_{pT\mu} = \left(\frac{1}{c_\ell^2} + c_{p,\ell}T\left(\frac{\rho_\ell-\rho_{\mathrm{g}}}{\rho_{\mathrm{g}}(h_\ell-h_{\mathrm{g}})} + \frac{\Gamma_\ell}{c_\ell^2}\right)^2\right)^{-\frac{1}{2}} \neq c_\ell. \tag{9.19}$$

This implies that when an infinitesimal amount of gas is added to a pure liquid, the mixture speed of sound will change drastically, and vice versa. The discontinuity in the single-phase limit may cause significant numerical challenges, and is not physically plausible, as pointed out by e.g. Städtke [26, Chap. 4]. It is interesting to note that only the combination of all three relaxation processes together causes this discontinuity, while any other combination does not exhibit such a behaviour.

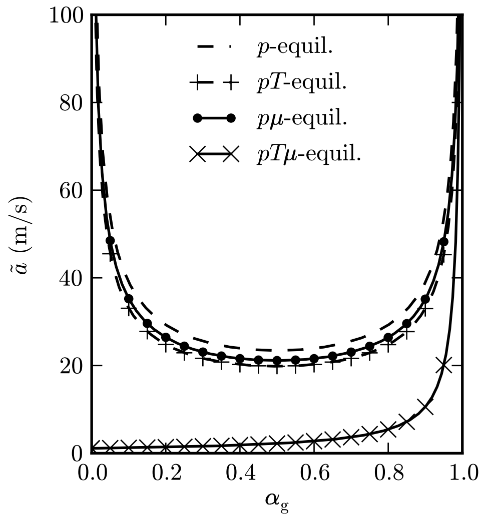
10. Speed of sound comparison. In this section, we will present plots illustrating the mixture speed of sound for water and carbon dioxide at industrially relevant conditions, illustrating the impact of the different equilibrium assumptions on the speed of sound. Plots with the same parameters were presented in Ref. [10] for five of the models, but in this section we complete the picture by considering all eight models in the hierarchy.

Figure 10.1a shows the mixture speed of sound in a two-phase water-steam mixture at atmospheric pressure, $p = 10^5$ Pa. The other parameters are shown in Table 10.1. We recognise that mechanical equilibrium has the most significant impact on the speed of sound, while thermal and chemical equilibrium assumptions have a much smaller effect. In Figure 10.1b, we take a closer look at the range 0–100 m/s. The full equilibrium model is, as expected, not continuous in the single-phase limit, clearly visible at $\alpha_{\mathrm{g}} = 0$, where the two-phase speed of sound is $\tilde{a}_{pT\mu} \approx 1$ m/s, whereas the liquid speed of sound is $c_\ell = 1543.4$ m/s.

The differences between the different models are perhaps even clearer in Figure 10.2, showing the speed of sound for a two-phase CO_2 mixture at $p = 50$ bar. The other parameters are listed in Table 10.2. In this figure, the subcharacteristic condition, predicting that the speed of sound is



(a) Full range



(b) Only 0 − 100 m/s

Figure 10.1: Mixture speed of sound in a water-steam mixture at atmospheric pressure.

Table 10.1: Parameters for a water-steam mixture at atmospheric pressure.

Quantity	Symbol	Unit	Gas	Liquid
Pressure	p	MPa	0.1	0.1
Temperature	T	K	372.76	372.76
Density	ρ	kg/m^3	0.59031	958.64
Speed of sound	c	m/s	472.05	1543.4
Heat capacity	c_p	J/kg K	2075.9	4216.1
Entropy	s	m^2/s^2 K	7358.8	1302.6
Grüneisen coefficient	Γ	(dimensionless)	0.33699	0.4

lowered for each imposed equilibrium assumption, is clearly illustrated. Once again, thermal and chemical equilibrium alone has little effect on the mixture speed of sound, and only combining the three equilibrium conditions leads to a discontinuous speed of sound in the single-phase limit.

For more discussions on models and experimental values for the speed of sound in two-phase systems, a number of works exist. Henry et al. [12] present experimental values for the speed of sound in different flow regimes in a water-steam system, while Kieffer [16] compares experimental values with certain models. Städtke [26] also discusses a variety of different of models and their speeds of sound. Furthermore, Zein et al. [30] have interesting discussions on how the speeds of the different relaxation processes typically are related.

11. Conclusion and further work. We have studied the complete hierarchy of averaged two-phase homogeneous flow models that arises by assuming equilibrium in different combinations of pressure, temperature and chemical potential, of which the T-, μ- and $T\mu$-equilibrium models represented original contributions. The models were formulated as hyperbolic relaxation systems with source terms accounting for heat, mass and volume transfer between the phases. Wave velocities for each model were derived, and we showed how the subcharacteristic condition leads to the requirement that the mixture speed of sound decreases when equilibrium assumptions are imposed. This requirement was explicitly and analytically shown using sums of squares. Furthermore, it was illustrated how the different equilibrium assumptions affect the speed of sound

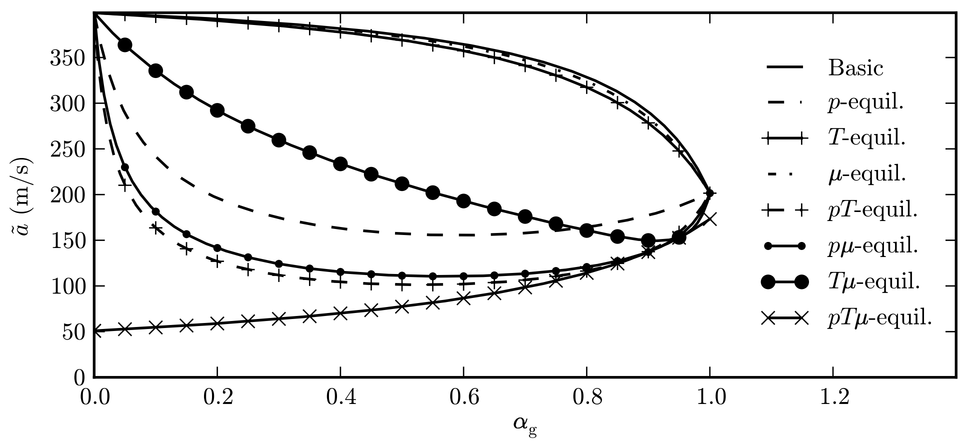


Figure 10.2: Mixture speed of sound in a two-phase CO_2 mixture at 50 bar.

Table 10.2: Parameters for a two-phase CO_2 mixture at 50 bar.

Quantity	Symbol	Unit	Gas	Liquid
Pressure	p	MPa	5.0	5.0
Temperature	T	K	287.43	287.43
Density	ρ	kg/m^3	156.71	827.21
Speed of sound	c	m/s	201.54	398.89
Heat capacity	c_p	J/kg K	3138.0	3356.9
Entropy	s	m^2/s^2 K	1753.9	1128.8
Grüneisen coefficient	Γ	(dimensionless)	0.30949	0.63175

in relevant cases for a water-steam mixture and two-phase carbon dioxide. We have also shown how the assumption of full equilibrium leads to a discontinuous speed of sound in the single-phase limit, a phenomenon which is quite unique for this model.

In further work, the hierarchy could possibly be extended to inhomogeneous flow models, i.e. different velocities for the two phases, formulated using two momentum equations and velocity relaxation.

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