Energy Norm Error Estimates

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Outline

- Residual based estimates
- Local problems
- Recovery estimates
- Lower bounds
- Refinement techniques
- Adaptive algorithms
- Other sources of error

Poisson Equation

Strong form. Find $u \in H_0^1(\Omega)$ such that

$$\begin{aligned} -\triangle u = f \quad & \text{in } \Omega, \\ u = 0 \text{ on } & \partial\Omega. \end{aligned}$$

where $f \in H^{-1}(\Omega)$, and Ω is the domain.
Weak form. Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \quad \text{for all } v \in H_0^1(\Omega).$$

Discretization

We introduce a finite dimensional space $V_h \subset H_0^1(\Omega)$.

- The mesh consists of elements $K \in \mathcal{K}$.
- $h_K = diam(K)$ and $h(x) = h_K$ for $x \in K$.
- The functions $v \in V_h$ are piecewise polynomials.

Finite element method. Find $U \in V_h$ such that

$$\int_\Omega \nabla U \cdot \nabla v \, dx = \int_\Omega f v \, dx \quad \text{for all } v \in V_h.$$

Galerkin Orthogonality

We subtract the finite element equations from the weak form and obtain the following equations for the error $e = u - U$,

$$\int_\Omega \nabla e \cdot \nabla v \, dx = 0 \quad \text{for all } v \in V_h,$$

i.e. the error e is orthogonal to the space V_h in this sense. We let πv be the interpolant of v.

Residual Based Estimate

$$\begin{aligned}
\|\nabla e\|^2 &= \int_\Omega \nabla e \cdot \nabla e \, dx \\
&= \int_\Omega \nabla e \cdot \nabla (e - \pi e) \, dx \\
&= -\int_\Omega \triangle e (e - \pi e) \, dx \\
&= \int_\Omega (f + \triangle U)(e - \pi e) \, dx
\end{aligned}$$

We need to interpret $\int_\Omega (f + \triangle U) v \, dx$.

Residual Based Estimate, cnt

$$\int_\Omega (f + \triangle U) v \, dx = \sum_{K \in \mathcal{K}} \int_K (f + \triangle U) v \, dx$$
$$- 0.5 \int_{\partial K \setminus \Gamma} [n \cdot \nabla U] v \, ds$$
$$\leq \sum_{K \in \mathcal{K}} \| h(f + \triangle U) \|_K \| h^{-1} v \|_K$$
$$+ 0.5 \| h^{1/2} [n \cdot \nabla U] \|_{\partial K \setminus \Gamma} \| h^{-1/2} v \|_{\partial K \setminus \Gamma}$$

we let $v = e - \pi e$.

Residual Based Estimate, cnt

Interpolation estimate.

$$\sum_{K \in \mathcal{K}} \|h^{-1}(e - \pi e)\|_K^2 \leq C\|e\|_1^2 \leq C\|\nabla e\|^2,$$

where the last inequality is due to Poincare-Friedrichs Lemma.

Trace inequality.

$$\|h^{-1/2}(e - \pi e)\|_{\partial K}^2 \leq C\|h^{-1}(e - \pi e)\|_K \|(e - \pi e)\|_{K,1}$$

$$\sum_{K \in \mathcal{K}} \|h^{-1/2}(e - \pi e)\|_{\partial K}^2 \leq C\|\nabla e\|^2$$

Residual Based Estimate, cnt

Remember

$$\|\nabla e\|^2 = \int_\Omega (f + \triangle U)(e - \pi e)\, dx$$

$$\leq \sum_{K \in \mathcal{K}} \frac{D}{2} \|h(f + \triangle U)\|_K^2 + \frac{1}{2D} \|h^{-1} e - \pi e\|_K^2$$

$$+ \frac{D}{2} \|0.5 * h^{1/2} [n \cdot \nabla U]\|_{\partial K \setminus \Gamma}^2 + \frac{1}{2D} \|h^{-1/2} e - \pi e\|_{\partial K \setminus \Gamma}^2$$

$$\leq \frac{D}{2} \sum_{K \in \mathcal{K}} \|h(f + \triangle U)\|_K^2 + \|0.5 * h^{1/2} [n \cdot \nabla U]\|_{\partial K \setminus \Gamma}^2$$

$$+ \frac{1}{2} \|\nabla e\|^2, \quad D = 2C$$

Residual Based Estimate, cnt

We end up with,

$$\|\nabla e\|^2 \leq C \sum_{K \in \mathcal{K}} R_K^2,$$

where $R_K^2 = h_K^2 \|f + \triangle U\|_K^2 + \frac{1}{4} h_K \|[n \cdot \nabla U]\|_{\partial K \setminus \Gamma}^2$

Residual Based Estimate, cnt

We can extend this result to the case of variable coefficient,

$$-\nabla \cdot a\nabla u = f$$

in this case we get

$$\|\sqrt{a}\nabla e\|^2 \leq C \sum_{K\in\mathcal{K}} R_K^2,$$

where

$$R_K^2 = h_K^2 \| \frac{1}{\sqrt{a}} f + \nabla \cdot a\nabla U\|_K^2 + \frac{1}{4} h_K \| \frac{1}{\sqrt{a}} [n \cdot a\nabla U]\|_{\partial K\setminus\Gamma}^2$$

Residual Based Estimate, cnt

- Simple to implement
- Correct h-dependence of the error
- Unknown constants appear

Local Problems on Stars

Next we turn to estimates on mesh stars.
Nochetto-Carstensen derives the following technique for the energy norm, let $(v, w) = \int_\Omega vw\,dx$,

$$\begin{aligned}\|\nabla e\|^2 &= (-\triangle e, e) \\ &= (f + \triangle U, e) \\ &= \sum_{i=1}^{n} (\varphi_i (f + \triangle U), e) \\ &= \sum_{i=1}^{n} (\varphi_i \nabla E_i, \nabla e)_{S_i}\end{aligned}$$

Local Problems on Stars, cnt

The functions $E_i \in H^1(S_i)$ are defined to solve,

$$(\varphi_i \nabla E_i, \nabla v)_{S_i} = (\varphi_i(f + \triangle U), v)_{S_i},$$

for all $v \in H^1(S_i)$. This equation is solvable since the kernel consists of $v = C$ and $(\varphi_i(f + \triangle U), C)_{S_i} = 0$ from Galerkin Orthogonality.

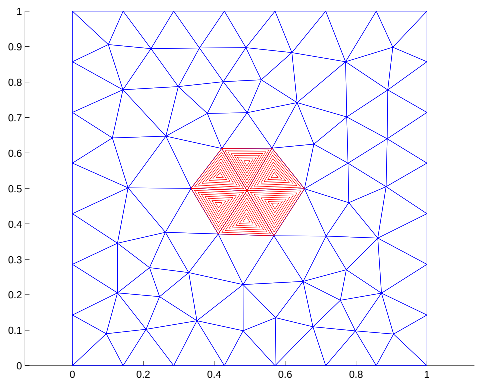


Figure 1: The mesh star S_i.

Local Problems on Stars, cnt

We get the following estimate,

$$\|\nabla e\|^2 = \sum_{i=1}^{n} (\varphi_i \nabla E_i, \nabla e)_{S_i} = (\sum_{i=1}^{n} \varphi_i \nabla E_i, \nabla e)$$

By using the Cauchy-Schwarz inequality we get,

$$\|\nabla e\|^2 \leq \| \sum_{i=1}^{n} \varphi_i \nabla E_i \|^2 = \sum_{K \in \mathcal{K}} \| \sum_{i=1}^{n} \varphi_i \nabla E_i \|_K^2$$

Local Problems on Stars, cnt

We perform a similar calculation to get a simpler version.

$$\|\nabla e\|^2 = (-\triangle e, e) = (f + \triangle U, e)$$
$$= \sum_{i=1}^{n} (\varphi_i (f + \triangle U), e) = \sum_{i=1}^{n} (\nabla E_i, \nabla e)_{S_i}$$

Here E_i is determined by,

$$(\nabla E_i, \nabla v)_{S_i} = (\varphi_i (f + \triangle U), v)_{S_i},$$

then we get, $\|\nabla e\|^2 \leq \sum_{K \in \mathcal{K}} \|\nabla \left(\sum_{i=1}^{n} E_i \right) \|_K^2$.

Local Problems on Stars, cnt

We can also solve the equations for E_i on stars of more layers.

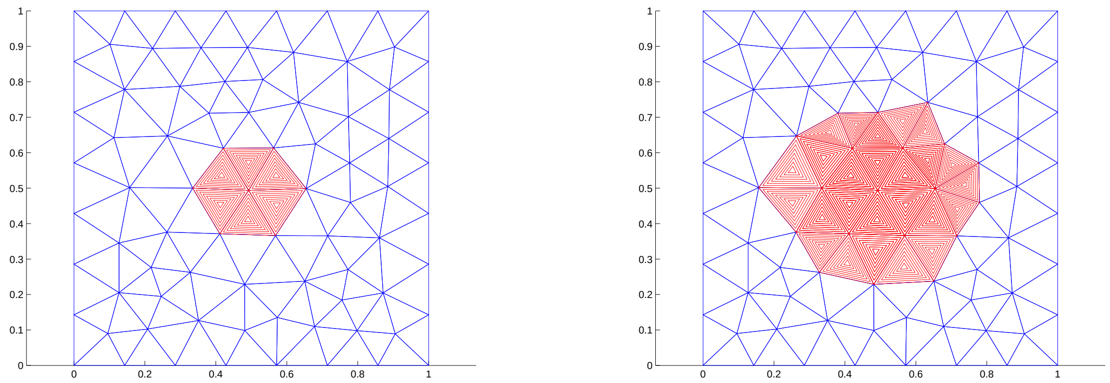


Figure 2: One and two layer mesh stars S_i^j $j = 1, 2$.

Local Problems on Elements

We now use Neumann conditions on the local problems,

$$\begin{aligned}\|\nabla e\|^2 &= (\nabla e, \nabla e) \\ &= \sum_{K \in \mathcal{K}} (\nabla e, \nabla e)_K \\ &= \sum_{K \in \mathcal{K}} (f + \triangle U, e)_K + (n \cdot \nabla e, e)_{\partial K} \\ &= \sum_{K \in \mathcal{K}} (f + \triangle U, e)_K + (\Sigma_n - n \cdot \nabla U, e)_{\partial K},\end{aligned}$$

where Σ_n is a conservative approximation of $n \cdot \nabla u$.

Local Problems on Elements

We define $E_K \in H^1(K)$ to solve,

$$(\nabla E_K, \nabla v)_K = (f + \triangle U, v)_K + (\Sigma_n - n \cdot \nabla U, v)_{\partial K},$$

for all $v \in H^1(K)$. This equation is solvable if

$$(f + \triangle U, C)_K + (\Sigma_n - n \cdot \nabla U, C)_{\partial K} = 0,$$

for a constant C i.e.

$$\int_K f + \int_{\partial K} \Sigma_n = 0.$$

We get the following estimate
$\|\nabla e\|^2 \leq \sum_{K \in \mathcal{K}} \|\nabla E_K\|^2$.

Local Problems on Elements

The Neumann approach requires

- Computation of equilibrium fluxes Σ_n
- Solution of local Neumann problems

Some references on computation of equilibrium fluxes

- Ainsworth-Oden
- Baker
- Ladeveze
- Larson-Niklasson

Local Problems

- No unknown constants
- More efficient
- More complicated
- E needs to be computed numerically
- We get a truth mesh error estimate

Recovery Estimates

Let $GU \in V_h^d$ be determined by the equation

$$(GU, v)_{Lump} = (\nabla U, v),$$

for all $v \in V_h^d$. Here $(\cdot, \cdot)_{Lump}$ refers to the lumped $L^2(\Omega)$-product. Then it holds,

$$\|\nabla e\| \leq C \sum_{K \in \mathcal{K}} \rho_K^2,$$

where $\rho_K = \|GU - \nabla U\|_K$.

Lower Bounds

Assume we compute an approximation of the error $E \in V_h \subset H_0^1(\Omega)$ such that,

$$(\nabla e, \nabla v) = (\nabla E, \nabla v),$$

for all $v \in V_h$. Then we have,

$$\begin{aligned} \|\nabla E\|^2 &= (\nabla E, \nabla E) \\ &= (\nabla e, \nabla E) \\ &\leq \|\nabla e\| \|\nabla E\|, \end{aligned}$$

So we get $\|\nabla E\| \leq \|\nabla e\|$.

Ex: Solve a local Dirichlet problem on each element.

Lower Bounds, cnt

For estimates based on local Neumann problems on stars we obtain an approximation E of the error e. E is discontinuous on element edges and satisfies,

$$(\nabla E, \nabla v) = (\nabla e, \nabla v),$$

for all $v \in H_0^1(\Omega)$ by construction.
Assume we can compute a continuous approx. E^c of E. Then we have

$$2\lambda(\nabla E, \nabla E^c) - \lambda^2 \|\nabla E^c\|^2 \leq \|\nabla e\|^2,$$

for all $\lambda \in \mathcal{R}$. See (Diez, Pares and Huerta).

Lower Bounds, cnt

$$
\begin{aligned}
0 &\leq (\nabla(e - \lambda E^c), \nabla(e - \lambda E^c)) \\
&= \|\nabla e\|^2 - 2\lambda(\nabla e, \nabla E^c) + \lambda^2 \|\nabla E^c\|^2 \\
&= \|\nabla e\|^2 - 2\lambda(\nabla E, \nabla E^c) + \lambda^2 \|\nabla E^c\|^2.
\end{aligned}
$$

This leads to

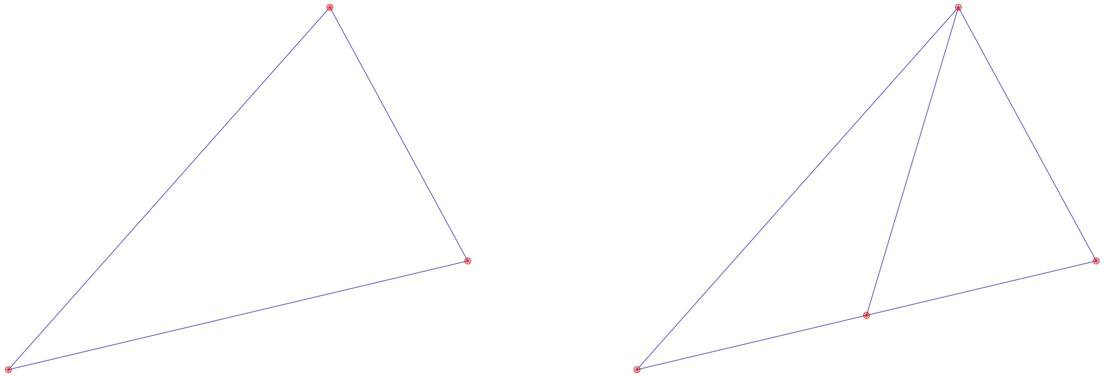
$$\|\nabla e\|^2 \geq 2\lambda(\nabla E, \nabla E^c) - \lambda \|\nabla E^c\|^2.$$

An optimal value $\lambda = \frac{(\nabla E, \nabla E^c)}{\|\nabla E^c\|^2}$ gives,

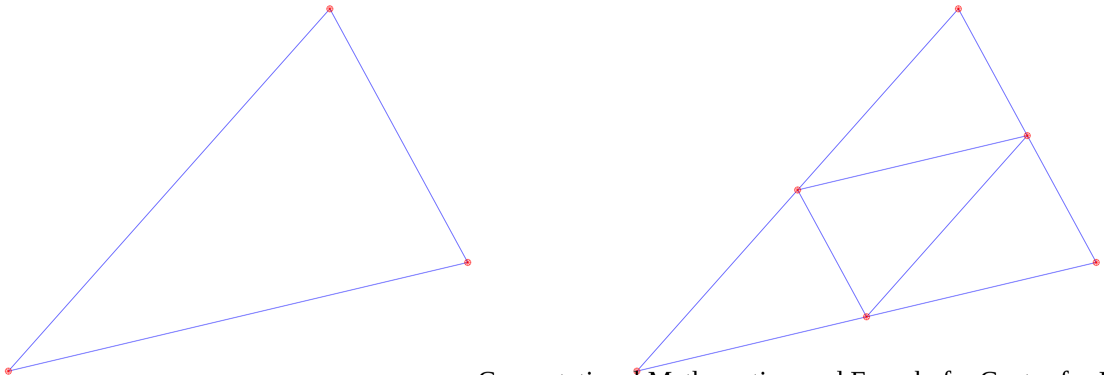
$$\|\nabla e\|^2 \geq \frac{|(\nabla E, \nabla E^c)|^2}{\|\nabla e\|^2}$$

Refinement Techniques

Bisect triangles. Joining the midpoint of the longest edge with the opposite vertex.

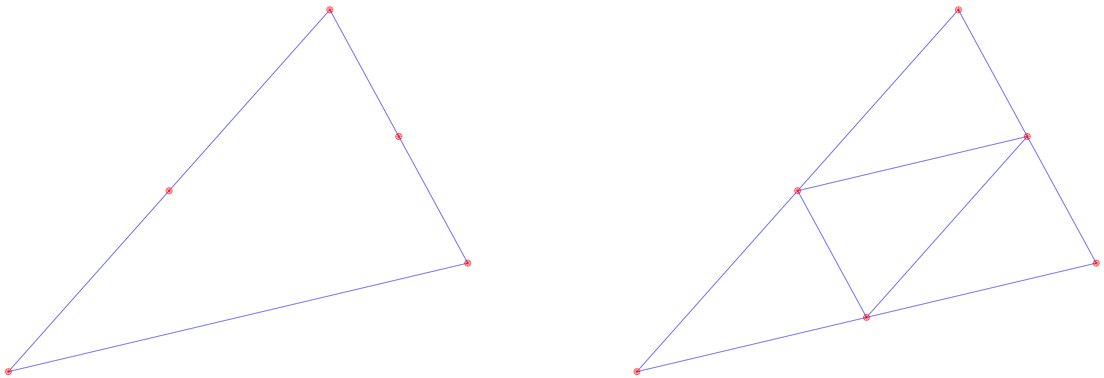


Divide into four parts.

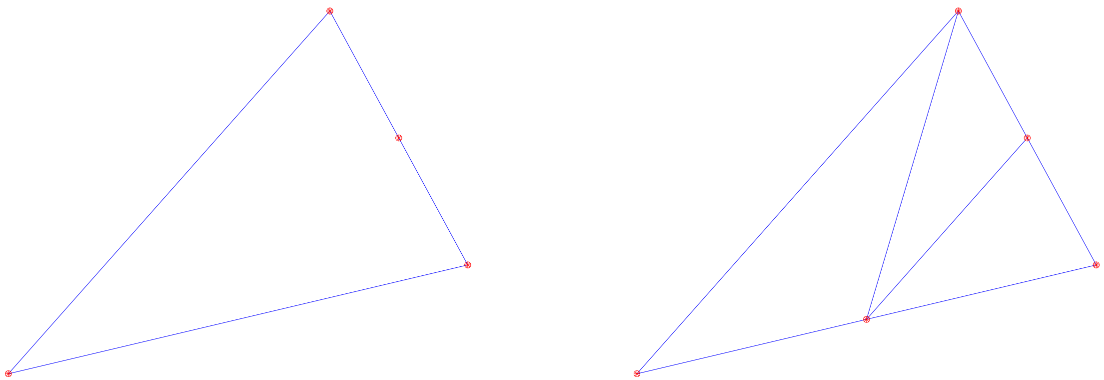


Refinement Techniques, cnt

Red refinement.

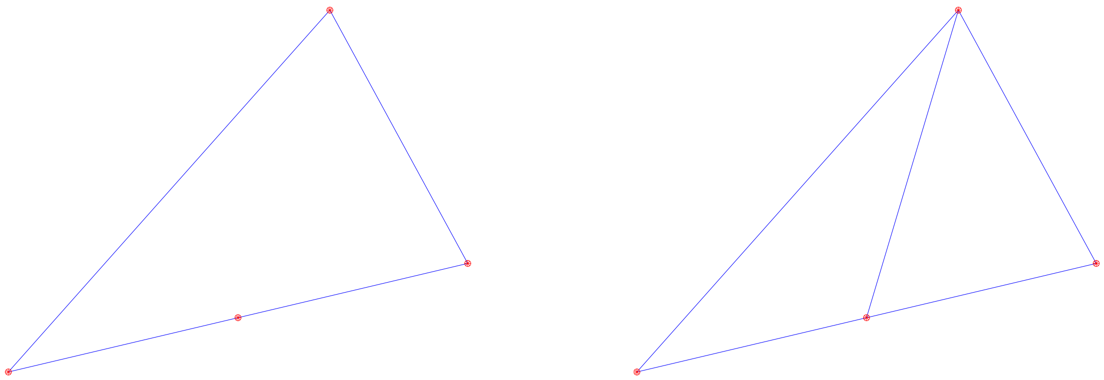


Blue refinement.



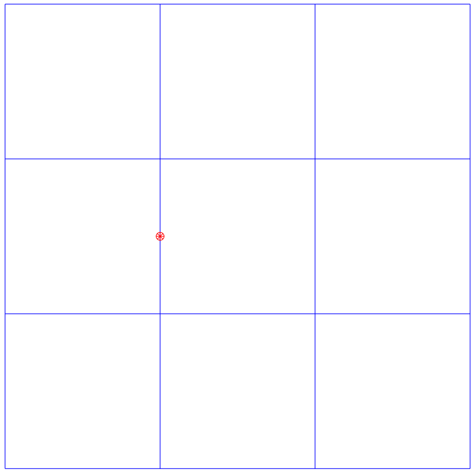
Refinement Techniques, cnt

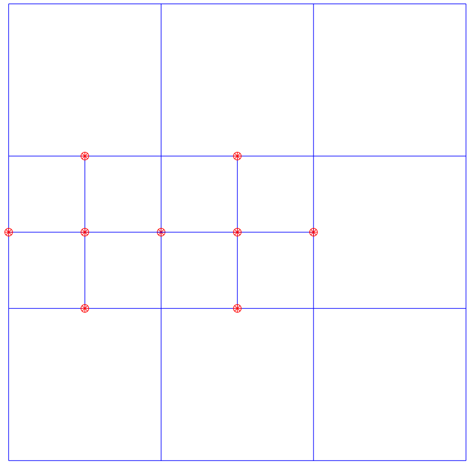
Green refinement.



Refinement Techniques, cnt

Bricks.





Hanging nodes may be dealt with,

- By enforcing strong continuity
- By enforcing weak continuity

Adaptive Algorithms

- Start on initial mesh
- Solve equations to get the solution U
- Calculate the error indicators E_K or R_K
- Refine elements K with large values of the indicator
- Go back to step 2 or stop if the error is small enough

In practice computer memory or time can also be used as a natural stopping criteria.

Adaptive Algorithms, cnt

There are different ways to choose which elements to refine. Assume that we have an estimate $\|\nabla e\|^2 \leq \sum_{K\in\mathcal{K}} \mathcal{R}_K^2$. Let $0 \leq \theta \leq 1$,

- Refine all elements where $\mathcal{R}_K \geq \theta \max \mathcal{R}_K$.
- Let $D \subset \mathcal{K}$ such that

$$\sum_{k\in D} \mathcal{R}_K^2 \geq \theta^2 \sum_{K\in\mathcal{K}} \mathcal{R}_K^2.$$

  Refine elements in D.

Other Sources of Error

A posteriori estimates can be extended to include

- Errors in data
- Errors in solution of algebraic system of equations
- Errors in solution due to approximation of geometry
- Quadrature errors

Error in Data

We study error in data f,

$$\begin{aligned}(\nabla e, \nabla e) &= (f + \triangle U, e) \\ &= (\tilde{f} + \triangle U, e) + (f - \tilde{f}, e) \\ &\leq |(\tilde{f} + \triangle U, e - \pi e)| + C\|f - \tilde{f}\|_{-1}\|e\|_1,\end{aligned}$$

so we get

$$\|\nabla e\|^2 \leq C \sum_{K \in \mathcal{K}} R_K^2 + C\|f - \tilde{f}\|_{-1}^2.$$

Algebraic Error

If the algebraic system of equation is solved approximately we get,

$$\|\nabla e\|^2 = (f + \triangle U, e - \pi e) + (f + \triangle U, \pi e),$$

where the second term is the algebraic residual.

Quadrature Error

Finally we consider quadrature error,

$$
\begin{aligned}
\|\nabla e\|^2 &= (f + \triangle U, e - \pi e) \\
&\quad + (\nabla e, \nabla \pi e) - (\nabla e, \nabla \pi e)_h \\
&= (f + \triangle U, e - \pi e) \\
&\quad + (f, \pi e) - (f, \pi e)_h \\
&\quad + (\nabla U, \nabla \pi e) - (\nabla U, \nabla \pi e)_h,
\end{aligned}
$$

where $(\cdot, \cdot)_h$ in the form obtained by quadrature.