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Inverse problems arising in connection with ECG recordings

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1

Keywords

- Heart
- Electrical activity
- Bidomain model
- ECG: electrocardiogram
- Potential at the heart surface
- Infarctions

2

Three lectures

1. The Bidomain model, Nielsen
2. The electrical potential at the surface of the heart, Nielsen
3. Identifying heart infarctions, Lysaker

3

4

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The Bidomain model

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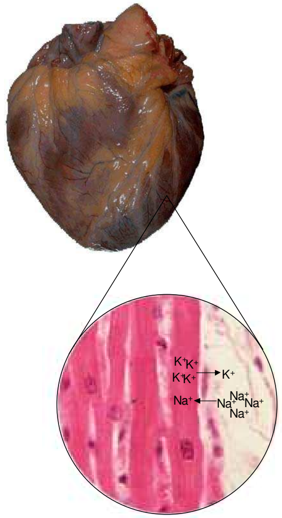
5

- Brief introduction to the Bidomain equations
- Electrical activity in the heart
- Motivation/Background
- The mathematical model

6



The heart

For each heartbeat, ions move in and out of the heart muscle cells in a complicated pattern.The most important ions are Na^+, K^+, Ca^{++}, and Cl^-.

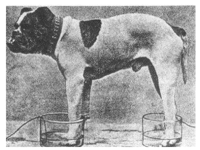
The movement of the ions cause the muscle fibers to contract and the heart to pump.

Normal electrical activity is important for the pumping function of the heart.

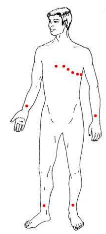
The electrical activity of the heart may be recorded on the surface of the body. The recording is called an electrocardiogram (ECG).

7

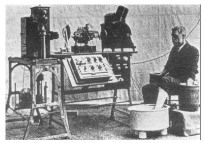
ECG



1887
The first ECG is recorded in London on Augustus Wallers dog Jimmy.



1943
The lead positions are standardized.

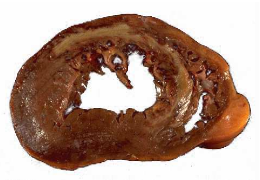


1911
A commercial ECG machine is constructed by Willem Einthoven.

2003
Worldwide, approximately 1 million ECGs are recorded every day.

8

Heart infarctions



20% of deaths in the western world are due to hearth infarctions and consequences thereof.

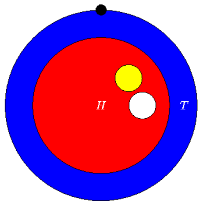
About 50% of patients admitted to surveillance units with acute chest pain suffer from a heart infarction.

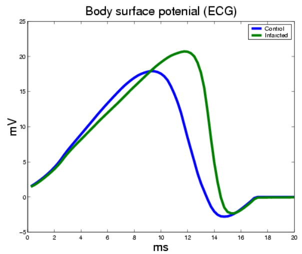
ECG is an important tool to diagnose heart infarction and heart cramp (angina).

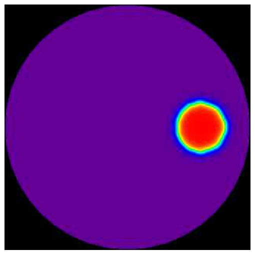
In some parts of the heart, the sensitivity of ECG is as low as 60%.

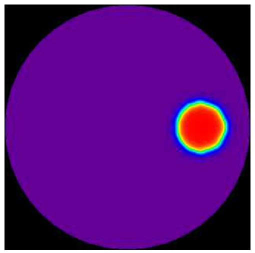
Normal interpretation of ECG only gives crude estimates: “large infarction” , “small infarction”.

9

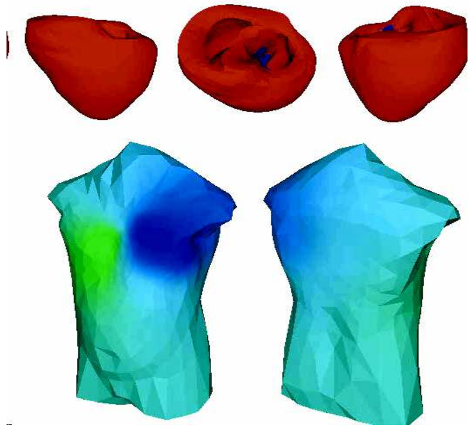






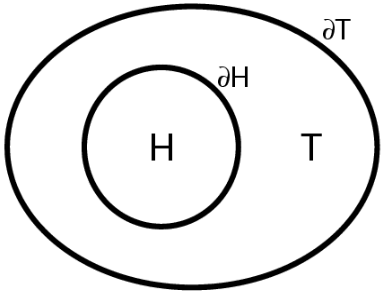


10



11

Computational domain sketched in 2D



12

Electrical potential in the torso

J~ current density

u~ electrical potential

M~ conductivity tensor

Conservation of current,

$$\nabla \cdot J = 0,$$

and Ohms law

$$J = -M\nabla u$$

gives

$$\nabla \cdot (M\nabla u) = 0 \quad \text{in } T$$

and, in addition

$$M \frac{\partial u}{\partial u} = 0 \quad \text{on } \partial T$$

Electrical potential in the heart

Intracellular (space within the cells)

J_i~ current density

u_i~ electrical potential

M_i~ conductivity tensor

q_i~ electrical charge

Extracellular (space outside the cells)

J_e~ current density

u_e~ electrical potential

M_e~ conductivity tensor

q_e~ electrical charge

Transmembrane potential

$$v = u_i - u_e$$

Conservation laws for intra and extra cellular domains:

$$\frac{\partial q_i}{\partial t} + \nabla \cdot J_i = -I_{\text{ion}}$$

$$\frac{\partial q_e}{\partial t} + \nabla \cdot J_e = I_{\text{ion}}$$

where I_{ion} models the ionic current.

From these two equations, the BiDomain model is derived

$$v_t + I_{\text{ion}}(v) = \nabla \cdot (M_i \nabla u_i)$$

$$0 = \nabla \cdot (M_i \nabla u_i) + \nabla \cdot (M_e \nabla u_e)$$

The BiDomain model was developed by Gesolowitz, Miller, Schmitt, and Tung in the early 70's. The equations have been studied by a series of researchers (Colli Franzone et al, Henriques, Trayanova et al, Huang et al, …)

Model for ionic current (I_{ion})

The appropriate model for the ionic current will depend upon the application.

In the simplest type it is just a function of *v*:

$$I_{\text{ion}} = f(v)$$

In more realistic models it depends on several factors: $I_{\text{ion}} = f(v, s)$

where *s* is a vector of variables including concentrations of ions and permeability of ion channels. These variables are typically governed by an ODE system: $\frac{\partial s}{\partial t} = F(v, s)$

in torso:

$$\nabla \cdot (M_T \nabla u_T) = 0$$

at torso boundary:

$$n \cdot (M_T \nabla u_T) = 0$$

heart boundary:

$$\begin{aligned} n \cdot (M_i \nabla (u_e + v)) &= 0 \\ n \cdot (M_e \nabla (u_e)) &= n \cdot (M_T \nabla (u_T)) \\ u_e &= u_T \end{aligned}$$

in heart:

$$\begin{aligned} C_m \chi \tfrac{\partial v}{\partial t} + \chi I_{ion} &= \nabla \cdot (M_i \nabla u_e) + \nabla \cdot (M_i \nabla v) \\ \nabla \cdot ((M_i + M_e) \nabla u_e) &= -\nabla \cdot (M_i \nabla v) \\ \tfrac{d[Na]}{dt} &= -\tfrac{I_{\text{Na,tot}} \cdot A_{\text{Cap}}}{V_{\text{Myo}} \cdot z_{\text{Na}} \cdot F} \\ \tfrac{d[\text{K}]}{dt} &= -\tfrac{I_{\text{K,tot}} \cdot A_{\text{Cap}}}{V_{\text{Myo}} \cdot z_{\text{K}} \cdot F} \\ \tfrac{d[\text{Ca}]}{dt} &= -\tfrac{I_{\text{Ca,tot}} \cdot A_{\text{Cap}}}{V_{\text{Myo}} \cdot z_{\text{Ca}} \cdot F} - (I_{\text{Up}} - I_{\text{Leak}}) \cdot \tfrac{V_{\text{NSR}}}{V_{\text{Myo}}} - I_{\text{Rel}} \cdot \tfrac{V_{\text{JSR}}}{V_{\text{Myo}}} \\ \tfrac{d[\text{Ca}_{\text{JSR}}]}{dt} &= -I_{\text{Rel}} + I_{\text{Tr}} \\ \tfrac{d[\text{Ca}_{\text{NSR}}]}{dt} &= I_{\text{Up}} - I_{\text{Leak}} + I_{\text{Tr}} \cdot \tfrac{V_{\text{JSR}}}{V_{\text{NSR}}} \\ \tfrac{dg}{dt} &= \alpha_g(v)(1-g) - \beta_g(v) g \quad \text{for} \quad g = m, h, j, x, d, f \end{aligned}$$

with gating coeffeicients:

$$\begin{aligned} \alpha_m(v) &= 0.32 \cdot (v + 47.13)/(1 - e^{-0.1 \cdot (v+47.13)}) \\ \beta_m(v) &= 0.08 \cdot e^{-v/11} \\ \alpha_h(v) &= \begin{cases} 0.135 \cdot e^{(80+v)/(-6.8)} & \text{if } v < -40 \\ 0 & \text{otherwise} \end{cases} \\ \beta_h(v) &= \begin{cases} 3.56 \cdot e^{0.079 \cdot v} + 3.1 \cdot 10^5 \cdot e^{0.35 \cdot v} & \text{if } v < -40 \\ 1/(0.13 \cdot (1 + e^{(v+10.66)/(-11.1)})) & \text{otherwise} \end{cases} \\ \alpha_j(v) &= \begin{cases} (-1.2714 \cdot 10^5 \cdot e^{0.2444 \cdot v} - 3.474 \cdot 10^{-5} \cdot e^{-0.04391 \cdot v}) \cdot \frac{v+37.78}{1+e^{0.311 \cdot (v+79.23)}} & \text{if } v < -40 \\ 0 & \text{otherwise} \end{cases} \\ \beta_j(v) &= \begin{cases} 0.1212 \cdot e^{-0.01052 \cdot v}/(1 + e^{-0.1378 \cdot (v+40.14)}) & \text{if } v < -40 \\ 0.3 \cdot e^{-2.535 \cdot 10^{-7} \cdot v}/(1 + e^{-0.1 \cdot (v+32)}) & \text{otherwise} \end{cases} \\ \alpha_x(v) &= 7.19 \cdot 10^{-5} \cdot (v + 30)/(1 - e^{-0.148 \cdot (v+30)}) \\ \beta_x(v) &= 1.31 \cdot 10^{-4} \cdot (v + 30)/(-1 + e^{0.0687 \cdot (v+30)}) \\ \alpha_d(v) &= (0.035 \cdot (v + 10))/(1 - e^{-(v+10)/6.24}) \\ \beta_d(v) &= \alpha_d \cdot e^{-(v+10)/6.24} \\ \alpha_f(v) &= f_\infty / f_\tau \\ \beta_f(v) &= (1 - f_\infty)/f\tau \end{aligned}$$

$$\begin{aligned} f_\infty(v) &= 1/(1 + e^{((v+35.06)/8.6)}) + 0.6/(1 + e^{(50-v)/20}) \\ f_\tau(v) &= 1/(0.0197 \cdot e^{-(0.0337 \cdot (v+10))^2} + 0.02) \end{aligned}$$

and currents given as

$$\begin{aligned} I_{ion} &= I_{\text{Na,tot}} + I_{\text{K,tot}} + I_{\text{Ca,tot}} \\ I_{\text{Na,tot}} &= I_{\text{Na}} + 3I_{\text{NaCa}} + 3I_{\text{NaK}} + I_{\text{ns,Na}} + I_{\text{Nab}} + I_{\text{CaNa}} \\ I_{\text{K,tot}} &= I_{\text{K}} + I_{\text{K1}} + I_{\text{Kp}} - 2I_{\text{NaK}} + I_{\text{ns,K}} + I_{\text{CaK}} \\ I_{\text{Ca,tot}} &= I_{\text{Ca}} - 2I_{\text{NaCa}} + I_{\text{p(Ca)}} + I_{\text{Ca,b}} \\ I_{\text{Na}} &= \overline{G}_{\text{Na}} \cdot m^3 \cdot h \cdot j \cdot (v - E_{\text{Na}}) \\ I_{\text{Ca}} &= d \cdot f \cdot f_{Ca} \cdot \overline{I}_{Ca} \\ I_{\text{CaNa}} &= d \cdot f \cdot f_{Ca} \cdot \overline{I}_{Ca\text{Na}} \\ I_{\text{Ca,K}} &= d \cdot f \cdot f_{Ca} \cdot \overline{I}_{CaK} \\ I_{\text{K}} &= \overline{G}_K \cdot X_i \cdot x^2 \cdot (v - E_K) \\ I_{\text{K1}} &= \overline{G}_{K1} \cdot K1_\infty \cdot (v - E_{K1}) \\ I_{\text{Kp}} &= \overline{G}_{Kp} \cdot Kp \cdot (v - E_{K1}) \\ I_{\text{NaCa}} &= k_{NaCa} \cdot (1/(K^3_{\text{m,Na}} + [\text{Na}]^3_o)) \\ I_{\text{NaK}} &= \overline{I}_{\text{Na}K} \cdot f_{\text{Na}K} \cdot (1/(1 + (K_{\text{m,Nai}}/[\text{Na}]_i)^{1.5})) \cdot ([K]_o/([K]_o + K_{m,Ko})) \\ I_{\text{ns,Na}} &= \overline{I}_{ns,\text{Na}}/[1 + (K_{m,ns(Ca)}/[Ca]_i)^3] \\ I_{\text{ns,K}} &= \overline{I}_{ns,K}/[1 + (K_{m,ns(Ca)}/[Ca]_i)^3] \\ I_{\text{p(Ca)}} &= \overline{I}_{p(Ca)} \cdot ([Ca]_i/(K_{m,p(Ca)} + [Ca]_i)), \ \overline{I}_{p(Ca)} = 1.15, \ K_{m,p(Ca)} = 0.5 \\ I_{\text{Ca,b}} &= \overline{G}_{Ca,b} \cdot (v - E_{Ca,N}) \\ I_{\text{Na,b}} &= \overline{G}_{\text{Na},b} \cdot (v - E_{\text{Na}}), \ \overline{G}_{\text{Na},b} = 0.00141 \\ I_{\text{Up}} &= \overline{I}_{\text{up}} \cdot \frac{[\text{Ca}]}{[\text{Ca}] + K_{m,\text{Up}}} \\ I_{\text{Leak}} &= K_{\text{Leak}} \cdot [\text{Ca}]_{\text{NSR}}, \quad K_{\text{Leak}} = \overline{I}_{\text{up}}/\overline{[\text{Ca}]}_{\text{NSR}} \\ I_{\text{Tr}} &= ([\text{Ca}]_{\text{NSR}} - [\text{Ca}]_{\text{JSR}})/\tau_{\text{Tr}} \\ I_{\text{Rel}} &= G_{\text{Rel}} \cdot ([\text{Ca}]_{\text{JSR}} - [\text{Ca}]) \\ G_{\text{Rel}} &= \begin{cases} \overline{G}_{\text{Rel}} \cdot \frac{\Delta[\text{Ca}]_2 - \Delta[\text{Ca}]_{th}}{K_{m,\text{Rel}} + \Delta[\text{Ca}]_2 - \Delta[\text{Ca}]_{th}} (1 - e^{-t/\tau_{on}}) \cdot e^{-t/\tau_{off}} & \text{if } \Delta[\text{Ca}]_2 > \Delta[\text{Ca}]_{th} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Cell models of ionic currents

- Hodgkin, Huxley
- Beeler, Reuter
- DiFrancesco
- Luo, Rudy
- Winslow et al

The BiDomain Model

$$\frac{\partial s}{\partial t} = F(v, s)$$

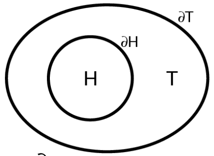
$$\chi C_m \frac{\partial v}{\partial t} + \chi I_{\text{ion}}(v, s) = \nabla \cdot (M_i \nabla v) + \nabla \cdot (M_i \nabla u_e)$$

$$0 = \nabla \cdot (M_i \nabla v) + \nabla \cdot ((M_i + M_e) \nabla u_e)$$

where

- u_e Extra cellular potential
- v Transmenbrane potential, $v = u_i - u_e$
- s State vector (concentrations and gates)
- F Rate functions
- I_{ion} Total ionic current
- C_m Membrane capacitance
- χ Membrane surface area to volume ratio
- M_i Intra cellular conductivity tensor
- M_e Extra cellular conductivity tensor

Boundary conditions



$$\partial H: \quad M_i \frac{\partial}{\partial n}(v + u_e) = 0$$

$$u_0 = u_T \ \ (u_0 \text{ is } u \text{ in } T)$$

$$M_e \frac{\partial u_e}{\partial n} - M_0 \frac{\partial u_0}{\partial n} = 0$$

$$\partial T: \quad M_0 \frac{\partial u_0}{\partial n} = 0$$

21

The BiDomain Model

$$\frac{\partial s}{\partial t} = F(v, s)$$

$$\chi C_m \frac{\partial v}{\partial t} + \chi I_{\text{ion}}(v, s) = \nabla \cdot (M_i \nabla v) + \nabla \cdot (M_i \nabla u_e)$$

$$0 = \nabla \cdot (M_i \nabla v) + \nabla \cdot ((M_i + M_e) \nabla u_e)$$

22