Improved gravity splitting for streamline and reordering methods

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June 30, 2010

Abstract

For heterogeneous reservoirs, fast, stable and accurate methods are hard to obtain. Large changes in the velocity field leads to severe time-step restrictions in explicit schemes or expensive time steps in implicit schemes. In the absence of gravity, the exact velocity field will be loop-free in the sense that there are no closed integral curves. A sequential splitting strategy with streamline or reordering methods utilize this feature to make fast transport solves for advection-dominated flow. For streamline methods, the absence of closed integral curves ensures that all streamlines have end points in wells. Likewise, this property implies that the Jacobi matrix of an implicit scheme with an upwind flux approximation can be reduced to a triangular matrix by a permutation, or reordering, of the unknowns.

For the above methods the effect of gravity is usually handled by operator splitting in the transport equation. Gravity will introduce rotation in the total velocity, which may yield closed integral curves. Even though the effect of gravity in many cases will be very small, it can limit the efficiency and the robustness of streamline and reorder methods. To overcome this problem, we split the total velocity field in a loop-free part without the effect of gravity, and a part driven by gravity only. This is done by solving the pressure equation two times with different right-hand sides.

We demonstrate that the new splitting strategy will increase the efficiency and applicability of reordering methods and make streamline tracing more robust.

Introduction

Numerical approximation of multiphase flow in heterogeneous reservoirs generally give rise to large systems of nonlinear equations that need to be solved in order to advance the solution forward in time. To a large extent, the success or failure of simulation development depend on the robustness and efficiency of the nonlinear solvers rather than the quality of the discretization. This has led to widespread use of fully implicit formulations which promise unconditional stability. In practical simulations, however, robust implementations of fully implicit schemes must limit the length of the time step, depending on the complexity of the grid, the geology, fluid physics, discretization scheme etc. With increasingly large and complex reservoir descriptions, there is a growing demand for faster yet stable and predictable simulation technology.

To achieve higher efficiency in reservoir simulation technology, solvers tend to exploit special features of the flow physics. The most flexible approach is based on sequential operator splitting of a total pressure formulation, with specialized solvers for pressure and transport.

In this paper, we consider specialized approaches to compute the transport of saturations. A well-known example of this approach is streamline method where the pressure and velocity is computed on a grid, were the transport of phases is computed along one-dimensional curves. The power of streamline method is that the streamlines change slowly with time compared to the dynamics of saturation fronts. For heterogeneous reservoirs dominated by advection, streamline technology have been shown to have great advantages over standard simulation technology (Baker, 2001; Thiele, 2005; Datta-Gupta and King, 2007). A different approach is to exploit the underlying direction of flow to construct nonlinear Gauss-Seidel type iterations for standard finite volume discretizations (Appleyard and Cheshire, 1982; Kwok and Tchelepi, 2007; Natvig and Lie, 2008a,b). Such accelerated finite-volume schemes have been shown to yield up to two order of magnitude reduction in run-time compared to standard methods.

These methods are based on two important assumption, that the characteristics of the hyperbolic system are always positive and that the vector field is associated with potential flow. Both assumptions break down when gravity is included. One will then in general have both positive and negative characteristics, and a rotational component in the velocity field. This leads to streamlines that form closed loops and large irreducible blocks in the nonlinear systems for finite-volume schemes (Natvig and Lie, 2008b). Even for heterogeneous reservoirs where the advection dominated, this can significantly deteriorate the efficiency of streamline and reordering methods even though the convective velocity component is orders of magnitude smaller than the advective part in most of the domain. In addition solenoidal part of the velocity field is moving with the fluid since it is governed by the density difference, this make a operator splitting approach attractive.

In this paper we propose operator splitting approach where we split the total velocity in two part, one rotation-free advective velocity field and one rotational convective velocity field that account for gravitational overturning. The standard gravitational splitting used in streamline simulators is to use the total velocity field for streamline tracing, and treat the gravity driven segregation separately (Bratvedt et al., 1996; Batycky et al., 1997). A similar type of splitting schemes have also been used to incorporate gravity together with diffusion into numerical methods designed to be highly efficient for advective dominated processes (H. Holden and Lie, 2000; Frøysa et al., 2000; Datta-Gupta and King, 2007)

Using the splitting scheme proposed in this paper, we can avoid the difficulties associated with a rotational velocity field in the dominating dynamics of the system. No significant additional restrictions for splitting time steps in the transport are introduced since the new terms are of the same order of magnitude as the segregation term. The cost of the method is one extra pressure solve with different right hand side, to compute the advective velocity field. However, since the global advective velocity will change much slower than the local convective part, different splitting time steps for each splitting step are appropriate. Additionally, the cost of a second pressure solve low for many types of linear solvers since preprocessing, preconditioning, or factorizations of the coefficient matrix may be reused.

Mathematical description

To keep the description as simple as possible, we consider incompressible flow of two immiscible fluids with different densities. This system may be written as a system of an elliptic equation for pressure and a first-order hyperbolic conservation law for the transport of phases,

$$\nabla \cdot \vec{v} = q, \qquad \vec{v} + \lambda \mathsf{K}[\nabla p - (\lambda_w \rho_w + \lambda_o \rho_o)\vec{g}] = 0 \tag{1}$$

$$\phi \frac{\partial s_w}{\partial t} + \nabla f_w(\boldsymbol{v} + \lambda_o(\rho_w - \rho_o)\mathsf{K}\vec{g}) = q_w. \tag{2}$$

where p is the fluid pressure, $\boldsymbol{v}$ is the total Darcy velocity, K and ϕ are the absolute permeability and porosity, respectively, ρ_i are the phase densities, λ_i denote phase mobilities and $\vec{g}$ is the acceleration of gravity. The total mobility, fractional flow and source terms are defined by $\lambda = \lambda_w + \lambda_o$, $f_w = \lambda_w/\lambda$, and $q = q_w + q_o$, respectively. Throughout this paper, we use no-flow conditions for each of the equations in (1).

We consider numerical schemes for (1) based on a sequential splitting of the system, where a standard two-point or Mimetic finite difference approximation is used to discretize the elliptic equations and a first-order finite-volume scheme is used to discretize the transport equation. Without gravity, this scheme may be written as an nonlinear equation

$$\phi(s_i^{n+1} - s_i^n) + f(s_i^{n+1}) \sum_j \max(v_{ij}, 0) + \sum_j f(s_j^{n+1}) \min(v_{ij}, 0) = q_i, \tag{3}$$

for each grid cell i in the domain. As has been shown in (Natvig and Lie, 2008a), this scheme may be written in vector notation by introducing the upwind flux matrix V such that

$$\boldsymbol{V} = \begin{cases} \min(v_{ij}, 0), & i \neq j \\ \sum_j \max(v_{ij}, 0) \end{cases} \tag{4}$$

As has been shown in (Natvig and Lie, 2008a), the upwind flux matrix $\boldsymbol{V}$ can be rearranged to (block) lower triangular form by a symmetric permutation of rows and columns using Tarjan's algorithm (Duff and Reid, 1978). This induce a triangular structure on the system of nonlinear equations that enable the use of nonlinear Gauss-Seidel approach where each nonlinear equation is solved separately (in sequence). The efficiency of the resulting scheme is comparable to streamline methods.

To take full advantage of this approach, the velocity field must be irrotational in the sense that there are no irreducible diagonal blocks in the inflow matrix. This ensure that all nonlinear equations (3) may be solved as scalar equations. This is equivalent to require that no streamlines reenter a grid cell.

When gravity is included in the model, the total velocity may no longer be irrotational. Furthermore, the characteristics of the transport equation are no longer strictly positive. To overcome this problem, we propose an operator splitting of the transport equation in three parts, corresponding to pure advection with no loops due to gravity and only positive characteristics, a convection step with only loops in the velocity field and a segregation step with zero total velocity. To be more precise, we solve the pressure equation with two different right-hand sides:

$$\nabla \cdot \vec{v}_{adv} = q, \qquad \vec{v}_{adv} + \lambda \mathsf{K} \nabla p = 0 \tag{5}$$

$$\nabla \cdot \vec{v} = q, \qquad \vec{v} + \lambda \mathsf{K}[\nabla p - (\lambda_w \rho_w + \lambda_o \rho_o)\vec{g}] = 0 \tag{6}$$

The advection velocity $\boldsymbol{v}_{adv}$ is obviously free of gravity-induced rotation. By solving the complete pressure equation, we may obtain the remaining $\vec{v}_{conv} = \vec{v} - \vec{v}_{adv}$. Alternatively, we may solve an

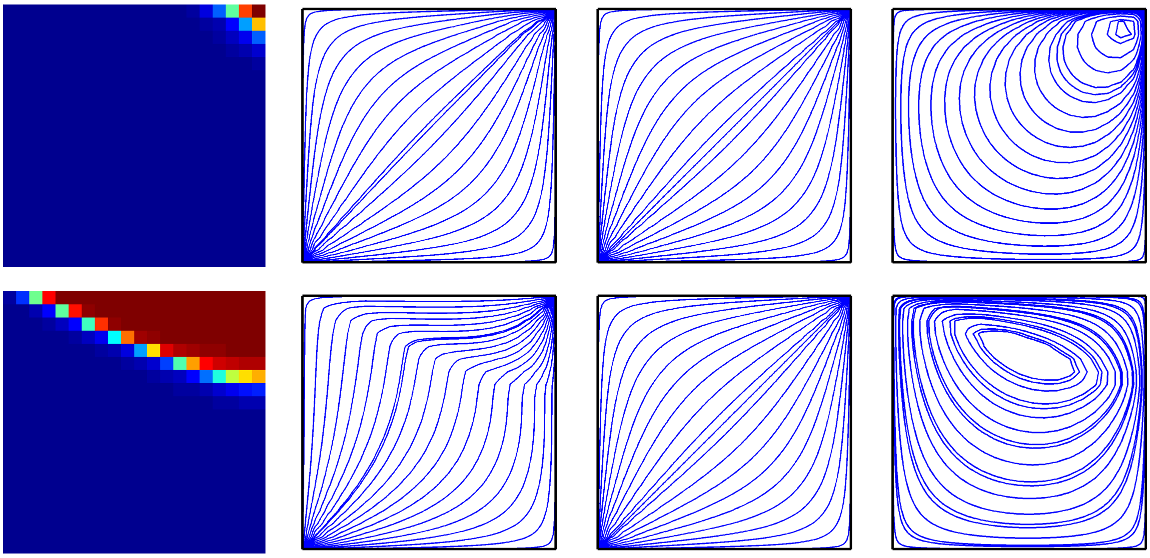

Figure 1 *Homogeneous domain with two fluids. Saturation (left), streamlines for the total velocity (middle left) convective velocity component (middle right) and streamlines for the advective velocity component (right) after* 1 *day (upper row) and after* 20 *days (lower row).*

equation for the convection field directly

$$\nabla \cdot \vec{v}_{conv} = 0 \quad \vec{v}_{conv} = -\lambda_t K(\nabla p_{conv} - \omega(S)\vec{g}). \tag{7}$$

It is worth noting that solving the pressure equation twice with different right-hand sides is not twice as expensive. We only have to generate the system matrix once. Furthermore, any preconditioning or factorization of the system matrix may be reused for the second pressure solve. It is also worth noting that the coupling between the pressure equation and the saturation equation trough can be quite severe in regions where the flow is dominated by gravitational effects. In these regions, the velocity field change as fast as the saturation fronts move. By splitting the velocity field in an advective part (irrotational) and a convective part (rotational), we can quantify the degree with which the gravitational effects induce tighter coupling in the operator splitting.

To advance the solution of (1) one time step, we propose the following operator splitting. First we solve (5) using a suitable solver. In this paper we will use either a two-point or mimetic finite difference scheme to discretize the pressure equation and a direct solver (Davis, 2004) to compute the solution. Then, the saturations are updated by solving three different transport equations for advection, convection and segregation:

$$\phi \frac{\partial s_w}{\partial t} + \nabla \cdot \Big(f_w \boldsymbol{v}_{adv} \Big) = q_w, \tag{8}$$

$$\phi \frac{\partial s_w}{\partial t} + \nabla \cdot \Big(f_w \boldsymbol{v}_{conv} \Big) = 0, \tag{9}$$

$$\phi \frac{\partial s_w}{\partial t} + \nabla \cdot \Big(f_w \big[(\lambda_o(\rho_w - \rho_o) \mathbf{K} \vec{g} \big] \Big) = 0. \tag{10}$$

In this paper, we will use a finite-volume discretization of (8)–(10). Since the characteristics of (8)–(9) are strictly non-negative, a simple upwind approximation (3) of the flux is sufficient. For (10), we use a mobility-weighted upwind flux approximation.

Example 1 *To illustrate the method, we computing the evolution of a two fluids with density* 100 *and* 1000 *Kg/m*^{3}*, linear relative permeability curves and viscosity* 1 *cP for both fluids. The domain is homogeneous with permeability* 100 *mD covering the unit square. The flow is driven by buoyancy forces,* *a positive source upper right corner injecting the lighter fluid at a rate of* $0.01m^3$*/day and a negative source in the lower left corner producing fluid at the same rate. Initially, saturation of the light fluid is zero in the whole domain. The injection of fluid result in a flow that driven by is a linear combination of buoyancy forces and a pressure gradient. The evolution of the velocity field and saturation field is computed using the proposed operator splitting.*

In Figure 1, the saturation is shown after 1 *day and after* 20 *days together with streamline plots of the Darcy velocity* $\boldsymbol{v}$*, the convective velocity field* $\boldsymbol{v}_{conv}$ *and the advective velocity field* $\boldsymbol{v}_{adv}$*. Is is clear the the gravity has a marked influence on the Darcy velocity when the saturation field changes. As expected, the convective velocity field is solenoidal and the advective velocity field is rotation-free and stationary. Using the proposed splitting isolates the advective part of the velocity field that changes slowly. The changes in the convective part of the velocity field take place on a different scale and with a different rate.*

For most cases a simple splitting step is sufficient were each of the three equations (8)–(10) are propagated to the end of the time step. A more accurate splitting could be used either between the equation (8) and (9) or for all of the equations. An simple strategy is to use a splitting with require just the amount of steps needed to get an accurate representation of the final step. As an example we should have that we need 100 steps for (8), 5 for (9) and 10 (10). Then a formal splitting sequence which would be expected to be optimal would be

$$S(t+T) = \left((U_{seg}(T/10))^2 \, U_{conv}(T/5) U_{adv}(T/100)^{10} \right)^5 S(t), \tag{11}$$

where $U(t)$ represent the time evolution operator for each term. The simplest approach which will be of relevant in advection dominated flow would be

$$S(t+T) = U_{seg}(T/n_{seg})^{n_{seg}} U_{conv}(T/n_{conv})^{n_{conv}} U_{adv}(T/n_{adv})^{n_{adv}} S(t) \tag{12}$$

Numerical experiments

In the first numerical example we show how a gravity dominated flow may affect streamline geometry and coupling of the unknowns in finite-volume schemes based on upwind flux approximations.

Example 2 *In the first example, we revisit the homogeneous porous square domain from Example 1. This time we reverse the flow direction, injecting the lighter fluid in the lower left corner and producing fluid form the upper right corner. The injection rate is reduced to* 0.001 m^3*/day to increase the influence of gravity.*

The purpose of this example is to show the impact the convective part of the velocity field may have on the efficiency of streamlines and single-point upwind schemes using reorder techniques to accelerate computations. The appearance of loops and spirals in the velocity field is an inconvenience when tracing streamlines since one often has to discard such streamlines. This makes the solution of transport along such streamlines in regions dominated by gravity unappealing. Single-point upwind schemes accelerated with optimal ordering of the grid cells is a promising technique to improve the effectiveness and robustness of implicit transport schemes. However, like for the streamline methods, the efficiency of optimal ordering is reduced when there are large loops in the velocity field. In fact, since optimal ordering is based on a reduction of the flux matrix v *in* (3) *to (block)-triangular form using a topological sorting algorithm (Natvig and Lie, 2008a,b; Duff and Reid, 1978), the algorithm is sensitive to small reverse fluxes that are caused by gravitational effects.*

On the left in Figure 2, the saturation of light fluid is shown after 20 *days in the upper plot and after* 100 *days in the lower plot. In this example, the advective flow rate that is imposed by the pair of sources is so small that the flow is almost fully segregated. A sharp interface forms between the fluids*

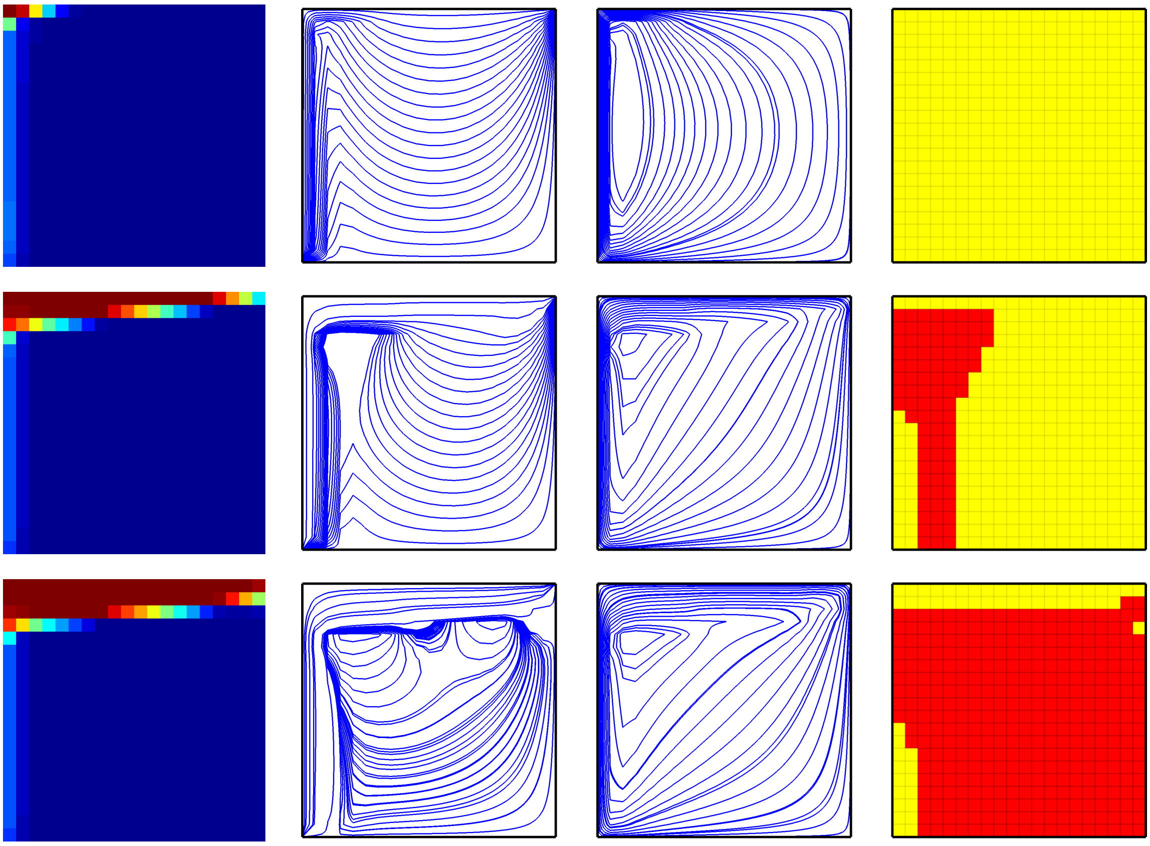

Figure 2 *Homogeneous domain with two fluids. Saturation (left), streamlines for the total velocity (middle) and grid cells associated with strongly connected components in the upwind flux matrix of* (3) *(right) after* 20 *days (top), after* 100 *days (middle row) and after* 300 *days (bottom).*

with a corresponding jump in density that yield strong convection loops. In Figure 2, we have plotted streamlines for the Darcy velocity and the convective velocity. The streamlines for the advective part are identical to the those plotted in Figure 1. It is clear that the effect of gravity dominates the flow in this example, producing loops in the Darcy velocity.

In the right column of Figure 2, we have plotted shown the grid, with grid cells associated with (large) irreducible diagonal blocks in the nonlinear equations for the single-point upwind scheme (3) *colored red. For an accurate solution of the system of nonlinear equations* (3)*, these grid cells must be solved simultaneously. The grid cells that are colored yellow are not coupled, which means that the saturation in these grid cells may be computed by solving the* (3) *independently of the other cells in the grid.*

We see that large irreducible blocks form when the light fluid accumulate at the top boundary. After 100 *days,* 89 *of* 400 *grid cells are connected in one irreducible diagonal block. This will significantly reduce the efficiency of optimal ordering. After* 300 *days, there are two irreducible blocks with* 341 *grid cells in one and* 4 *in the other.*

To emphasize that the gravity splitting we propose is applicable to realistic reservoirs with complex, twisted and deformed grid cells, faults, thin layers, barriers etc. we report preliminary results for gravitational splitting on a real reservoir model.

Example 3 *The model in this example is based on a real field model from the North Sea with rough geometry and geological description that has about* 40 000 *grid cells. The original reservoir geometry and geology is used, but we have modified the fluids and wells in this example. We have filled the original gas cap with oil to make the model more suitable for an incompressible formulation. Initially,*

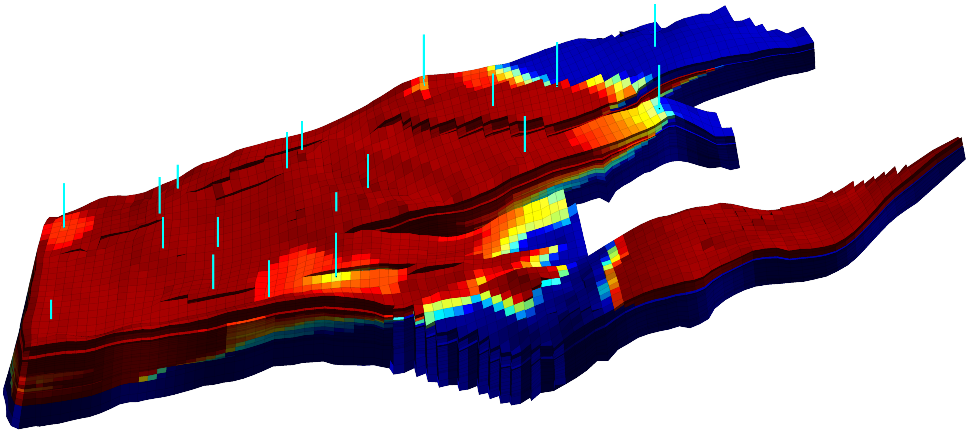

Figure 3 *Realistic reservoir model from North Sea. The plot shows the saturation field after* 10 *year of water injection with pressure time steps of* 2.5 *years.*

the reservoir is in near hydrostatic equilibrium with the original oil-water contact preserved. We inject water in some of the original wells. Furthermore, we use Corey relative permeability curves with Corey exponent 4*, viscosity* 0.318 *cP and* 1 *cP. The density difference used is* 174 *kg. In Figure 3, we show the water saturation after* 10 *years of injection.*

The purpose of this example is to report the time scales associated with advection, convection and segregation for a realistic geological model. Table 1 show the CFL time step restrictions for the advection step (8)*, the convection step* (9)*, and the segregation step* (10)*. We see that the advection step has the most severe time step restriction. The time step restriction for the segregation step is almost one order of magnitude less severe, whereas the convection step is more than two orders of magnitude less restrictive. We also see that the CFL time of the convection step stabilize on the order of* 200 *days for longer times.*

For the segregation an explicit method would be feasible since the time step is 6 *days. The value of* 1 *day for the advection step is on the border of being a reasonable time step for an explicit method. This model is coarse in particular around the wells. A further lateral refinement around the well would further increase the time step restriction of the advection step but not effect the gravity terms significantly.*

As a measure of complexity of the velocity fields we report the some statistics on the loops, that appear in the discrete velocity field, or more precisely, the number and size of the irreducible diagonal blocks in the upwind flux matrix of (3)*. We identify the irreducible blocks by permuting the upwind flux matrix to block diagonal form using Tarjan's algorithm (Duff and Reid, 1978).*

Tables 2 and 3 show how the number of loops with cells more than one, number of cells in the largest loop and the number of cells in loops with cells larger than one. This quantities give a good picture of the difficulties of the nonlinear problem which would ultimately limit the efficiency of implicit methods for long time steps. The advection calculated with the TPFA method is completely reordered, while the total velocity have about 500 *cells involved in loops. The convection part has loops covering almost the complete domain.*

Numerical method which have the reorder property for advection is highly favorable when using reordering or streamlines. So far, we have only used a two-point flux approximation to compute the total pressure and velocity. It is well-known that the two-point approximation is ill-suited for rough geometry or strongly anisotropic permeability tensors. If one use a convergent scheme, such as the mimetic finite-difference method, to compute the pressure and velocity one invariably gets loops velocity field, even without gravity. There seem to be a an intimate relation between loops in the velocity field and monotonicity of the pressure, see Figure 7 in Lie et al. (2010). Unfortunately there exist no (conver-

Time (years)	advection (days)	convection (days)	segregation (days)
0.25	1	46	6
0.5	1	79	6
0.75	1	82	6
1	1	85	6
1.25	1	87	6
2.5	1	120	6
5	1	200	6
7.5	1	213	6

Table 1 *Estimated time step restrictions for each of the splitting steps* (8)–(10) *in Example 3 at different simulation times.*

Time	convection			convection+advection		
	N	max	#cells	N	max	#cells
2.5	26	37835	38133	18	188	410
5	27	38912	39390	10	178	356
7.5	13	39913	40820	15	186	404

Table 2 *Number of loops N, maximal loop size (max) and the total number of grid cells in loops for the model in Example 3. The pressure is computed using a two-point flux approximation with a time step of* 2.5 *years.*

gent) monotone method with local stencil that guarantee monotone pressure (Keilegavlen et al., 2009). Likewise, our experience is that no convergent method guarantee loop-free velocity fields. In Table 4 we have counted the number of loops for the advection, convection and total Darcy velocity fields. In this case, the number of loops produced by the mimetic scheme is about the same as the the number of loops produce produced by gravity.

To understand the above result it is useful to estimate the time steps restriction of the different terms. It give us the relative strength and importance of the three different terms we have divided the transport into. We start by the segregation where the CFL time step is given by.

$$t^{seg} \sim \Delta z / \Big(K_z g \Delta\rho \frac{d}{ds} [\lambda_o f_w] \phi \Big) \tag{13}$$

The convection step is dependent on the particular distribution of the fluid since this is decided by the gravity driven total velocity. However a useful estimate can be derived based on the Dupuit approximation which are used in vertical equilibrium models (Bear, 1988; Martin, 1958; Coats et al., 1971). Assuming a thin layer of height h of the light fluid at the top of the aquifer gives the CFL time step due

Time (years)	convection			convection+advection		
	N	max	#cells	N	max	#cells
0.25	9	31709	32489	34	443	728
1	8	36763	36920	22	202	594
2	9	37856	38002	19	195	563
2.25	9	38092	38246	20	193	570

Table 3 *Number of loops N, maximal loop size (max) and the total number of grid cells in loops the model in Example 3. The pressure is computed using a two-point flux approximation with a time step of* 0.25 *years.*

Time (years)	advection			convection			convection+advection		
	N	max	#cells	N	max	#cells	N	max	#cells
0.25	424	3314	9038	98	33451	35314	453	2245	8697
0.50	433	3268	9514	76	36277	37720	423	3274	10272

Table 4 *Number of loops N, maximal loop size (max) and number of cells in loops for the model in Example 3. The pressure is computed using the mimetic finite-difference method with a time step of* 0.25 *years.*

to motion in the x direction,

$$t^{conv,x} \sim \Delta x / \Big(K_x g \frac{\Delta\rho}{\mu} \frac{\partial h}{\partial x} \phi \Big). \tag{14}$$

Here, the μ is the viscosity of the light fluid, K_x is the permeability in the x direction, $\Delta\rho$ is the density difference and g the gravity. We have assumed a horizontal top layer. From the same equation we can find the CFL time step associated with vertical flow for the same term

$$t^{conv,z} \sim \Delta z / \Big(K_x g \frac{\Delta\rho}{2\mu} \frac{\partial^2 h^2}{\partial^2 x} \phi \Big). \tag{15}$$

The ratio between this to estimate is

$$t^{conv,x} / t^{conv,z} \sim \Delta x \frac{\partial^2 h^2}{\partial^2 x} / \Big(2\Delta z \frac{\partial h}{\partial x} \Big) \sim \Delta x H_z / (\Delta z L_x). \tag{16}$$

H_z denote a typical vertical length scale while L_x denote a horizontal length-scale an K_z is the permeability in the z direction. For most reservoir models where the number of grid cells used to resolve the lateral variation are larger than in the vertical direction, this ratio would be grater than one. We also notice that this ratio does not depend on K_z. The ratio of the segregation time step to the convection time step is

$$t^{seg} / t^{conv,x} \sim K_x \Delta z \frac{\partial h}{\partial x} / \Big(K_z \Delta x \frac{d}{ds} [\lambda_o f_w] \mu \Big) \sim \Delta z K_x H_z / (\Delta x K_z L_x). \tag{17}$$

If we assume $K_x/K_z = 10$, $\Delta x/\Delta z = 50 - 100$ and $H/L \sim 100$, we get a ratio between segregation and convection time steps, $t^{seg}/t^{conv,x}$, of $1/5$–$1/10$. Table 1 of Example 3, we observe that $t^{seg}/t^{conv,x} \sim 1/10$ or less, which suggest that the convection velocity for many models can be treated explicitly. When this is the case, the convection step will be relatively cheap to compute compared to the advection and segregation steps.

Conclusion

We have devised a new splitting method where we separate the rotation-free advective transport from the dynamics due to gravity. With the proposed method, it may be possibility to improve gravity splitting schemes that use methods fine-tuned for advective flow, such as streamline methods or reorder methods. In our numerical experiments, we have shown that the proposed splitting isolates the dynamics associated with gravity in an efficient manner.

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