On the Stokes–Brinkman Equations for Modeling Flow in Carbonate Reservoirs

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July 15, 2010

Abstract

Cavities and fractures can significantly affect the flow paths of carbonate reservoirs and should be accurately accounted for during flow simulation. Herein, our goal is to compute the effective permeability of rock samples based on high-resolution 3D CT-scans containing millions of voxels. To this end, we need a flow model that properly accounts for the effects of Darcy flow in the porous material and Stokes flow in the void volumes on all relevant scales. The presence of different length scales and large contrasts in the petrophysical parameters leads to highly ill-conditioned linear systems that make such a flow model very difficult to solve, even on large-scale parallel computers. To identify simplifications that render the problem computationally tractable, we analyze the relative importance of the Stokes and Darcy terms for a wide variety of parameter ranges on an idealized 2D model. We find that a system with a through-going free flow region surrounded by a low permeable matrix can be accurately modeled by ignoring the Darcy matrix and simulating only the Stokes flow. Using the obtained insight, we are able to compute the effective permeability of a specific model from a CT-scan that contains more than eight million voxels.

Introduction

Numerical simulation of flow and transport in carbonate reservoirs is a challenging problem. Carbonate reservoirs are composed of porous material that can contain cavities and fractures that are connected on multiple scales throughout the rock formation. The presence of these, often relatively large, void spaces can significantly alter the flow paths in the medium and should therefore be accurately accounted for in the flow model. Herein, we use flow simulations to compute the effective permeability of a small-scale representative elementary volume (REV) obtained from a volumetric CT-scan data. The purpose is to populate large-scale geological models with the computed effective petrophysical parameters.

The standard approach to compute the effective permeability is to solve a single-phase flow problem with a unit pressure drop applied in each axial direction of the rock sample. If the rock sample consists of only porous material, the flow inside the REV is described using the Darcy equations. On the other hand, if the REV also contains void spaces with free flow, a more sophisticated flow model may be needed. To this end, there are two main approaches. The first approach is to use Darcy's law in the porous regions and the Stokes equations in the voids, coupled with the Beavers–Joseph–Saffman conditions on the interfaces (Beavers and Joseph, 1967; Arbogast and Lehr, 2006; Arbogast and Gomez, 2009; Arbogast and Brunson, 2007). Alternatively, Popov et al. (2009b,a) recently proposed that the Stokes–Brinkman equations (Brinkman, 1947) is a good model for single-phase, incompressible flow in carbonate karst reservoirs.

The Stokes–Brinkman equations read,

$$\nabla p = -\mu \mathbf{K}^{-1}\vec{v} + \nabla \cdot \mu^*(\nabla\vec{v} + \nabla\vec{v}^T), \qquad \nabla \cdot \vec{v} = 0. \tag{1}$$

Here, p is the fluid pressure, $\vec{v}$ is fluid velocity, $\mathbf{K}$ is the permeability tensor, μ is the fluid viscosity, and μ^* is the so-called *effective viscosity* of the fluid. In general, it is agreed that the value of μ^* depends on the properties of the porous medium. It has been shown that the effective viscosity can be either smaller, or greater than the fluid viscosity μ (see Liu and Masliyah (2005) for details). Given large variations in the material properties, μ^* needs not be homogeneous. However, in practice it is often assumed that the effective viscosity is homogeneous, and that $\mu = \mu^*$. By choosing appropriate parameters, one can study the end-member cases of Darcy flow and Stokes flow: for infinite permeability we obtain the Stokes law, for $\mu^* = 0$ we have the Darcy law. Henceforth, we will assume (1) as our basic flow model.

Irrespective of the choice of flow model, it turns out that solving the flow problem numerically on high-resolution models from 3D CT-scans is a challenging task: the particular model considered herein consists of more than eight million voxels. Solving the Darcy–Stokes or Stokes–Brinkman equations requires higher-order methods (Popov et al., 2009a; Arbogast and Gomez, 2009; Karper et al., 2009; Willems, 2009), which on realistic 3D CT-scan can lead to discrete systems with 10^8–10^9 degrees of freedom that are highly ill-conditioned because the observed length scales and material properties in the REV models differ by many orders of magnitude (Arbogast and Gomez, 2009).

Alternative choices would be to use a two-step upscaling procedure or a multiscale method (Hou and Wu, 1997; Arbogast and Bryant, 2002; Chen and Hou, 2003; Jenny et al., 2003), e.g., as applied to the Stokes–Brinkmann equations by Gulbransen et al. (2010) and Willems (2009). A multiscale method is conceptually well-suited for modeling problems that involve different scales, as it reduces the complexity of a problem while preserving the fine-scale details. Multiscale methods have proved to be more robust than standard upscaling methods for Darcy flow (Kippe et al., 2008), but have to date not been widely applied to non-Darcy problems. After running several smaller examples using the multiscale Stokes–Brinkman method by Gulbransen et al. (2010) we found that several improvements were necessary in order to apply this method for real 3D cases. The needed improvements are discussed in more detail later on. In brief, although the method is promising, our current implementation is too computationally costly and immature for a model of this size and complexity. On the other hand, a two-stage upscaling

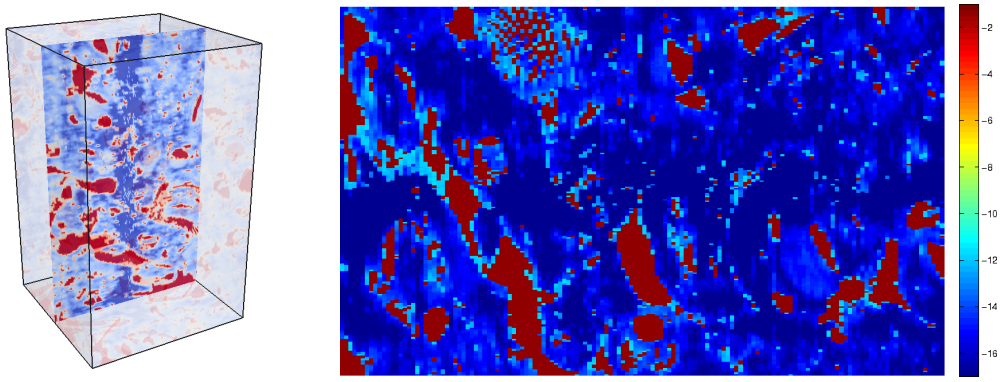

Figure 1 *Permeability of for the 3D model given on a regular grid with* $240 \times 240 \times 144$ *cells. The left plot shows the 3D permeability with a 2D cut through the domain. Here, red color denotes the void spaces (vugs). The right plot shows a 2D cut with permeability plotted on a logarithmic scale.*

procedure is not applicable either, as it would typically smear the free-flow regions or loose fine-scale connections and therefore produce too high or too low effective permeability.

In the following, we therefore take an alternative approach and identify simplifications that will render the fine-scale flow problem computationally tractable. To this end, we analyze an idealized 2D model to determine the parameters for which the full Stokes–Brinkman equations are necessary, and the parameter ranges for which the effective permeability can be computed by a Stokes (or Darcy) flow solver alone.

Model of a Rock Sample

As an example of a typical rock model, we will use a specific model that was obtained from a CT-scan of a $13 \times 13 \times 21$ cm rock sample (Zhang et al. (2004)). The resolution of the CT-scan was roughly $0.5 \times 0.5 \times 1.5$ mm, giving a model that consists of $240 \times 240 \times 144$ regular voxels. The values in the CT-image were converted to porosity based on the density of the rock and void spaces were identified as regions with low or zero density. The permeability of the matrix was computed using an upscaling formula by Jennings and Lucia (2003). Minimum matrix permeability is 10^{-4} mD and maximum is 10^3 mD.

The model is visualized in Figure 1, which shows the permeability; the red zones are void spaces (vugs) in which we expect free flow. From the 2D slice, one may get the impression that the vugs are not connecting one side of the model to the other, while in fact they are. Figure 2 shows a volume rendering of the three-dimensional structure of the void spaces inside the rock sample (in gray) with the borders between void space and porous material plotted in a yellowish color. The figure explains why all our attempts to perform a two-step upscaling procedure proved futile: after partitioning the whole REV into a set of non-overlapping and smaller REVs and then performing analytical or flow-based upscaling, the void spaces will be averaged out over the intermediate REVs and/or the connection between the outsides of the REV through void space will be lost. Likewise, Zhang et al. (2004) experienced that the connection of the vugs was lost when the 1/3-power averaging (Renard and de Marsily, 1997) was applied to upscale the permeability for the full REV.

As we will see later, the connection through void space is the most important factor for determining the effective permeability of this particular sample, and will be the key to developing an effective strategy for solving flow problems inside the REV. However, before we give any more details in this direction, we will briefly discuss numerical strategies for solving the Stokes–Brinkman equations.

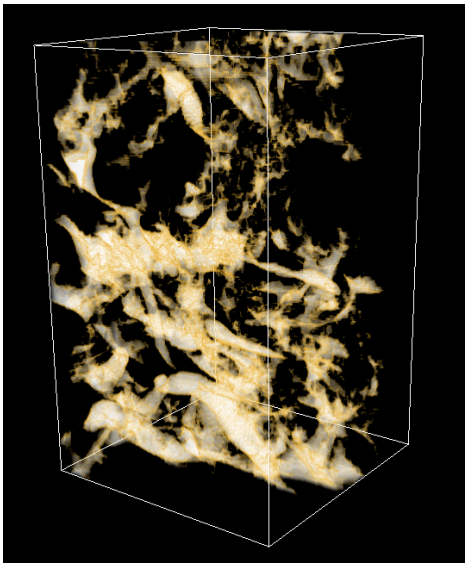

Figure 2 *Volume rendering of the 3D channels connecting the top and bottom of the model. Gray color shows the voids volumes and the void borders are shown in a yellowish color. To simplify the visualization, we have filtered out all isolated void spaces that are completely surrounded by the porous matrix, so that the figure only shows the vugs that are connected to the top and bottom of the sample.*

Numerical Solution of the Stokes–Brinkman Equations

To solve the Stokes–Brinkman equations (1) on the fine scale, we use a mixed finite-element formulation with distinct degrees of freedom defined for velocity and pressure variables. In 2D, we use the seven-node Crouzeix–Raviart triangle elements with quadratic velocity shape functions enhanced by a cubic bubble function, and a nine-node quad with quadratic velocity field. The pressure field is discontinuous, with linear shape functions. These elements have a number of advantages over other elements used for the Stokes and Stokes–Brinkman equations, e.g., the Taylor–Hood element used by Popov et al. (2009a); Gulbransen et al. (2010). The incompressibility constraint in (1) is implemented using the penalty method, i.e., the right-hand side of the divergence equation is set to p/κ. Then, incompressibility is assured by choosing a low penalty parameter κ and using the Powell and Hestenes iterations. The 2D fine-scale solver is implemented in the modified MILAMIN package (Dabrowski et al., 2008), which is an efficient finite-element solver in Matlab, capable of setting up, solving, and post-processing two-dimensional problems with one million unknowns in one minute on a standard desktop PC. For three-dimensional models, we do not use the penalty method, but rather include the pressure degrees of freedom in the system matrix. The resulting system of equations is symmetric and indefinite (zeros on the diagonal) and to solve it, we use the MINRES solver (Elman et al., 2005), preconditioned using the block preconditioner.

For the multiscale simulations, we use the multiscale mixed finite-element (MsMFE) method described in Gulbransen et al. (2010). This method uses a standard Darcy model to approximate pressure and fluxes on a coarse grid, whereas fine-scale effects are captured through basis functions that are computed numerically by solving local Stokes–Brinkman flow problems on the underlying fine scale grid. The local flow problems are solved using Taylor–Hood elements $\mathbb{Q}_2/\mathbb{Q}_1$, which are a standard set of stable elements for the Stokes equations.

Multiscale simulations

Our original idea was to use multiscale Stokes–Brinkman to perform simulations on the full 3D model. But as mentioned in the introduction, it soon became clear that the current implementation (Gulbransen

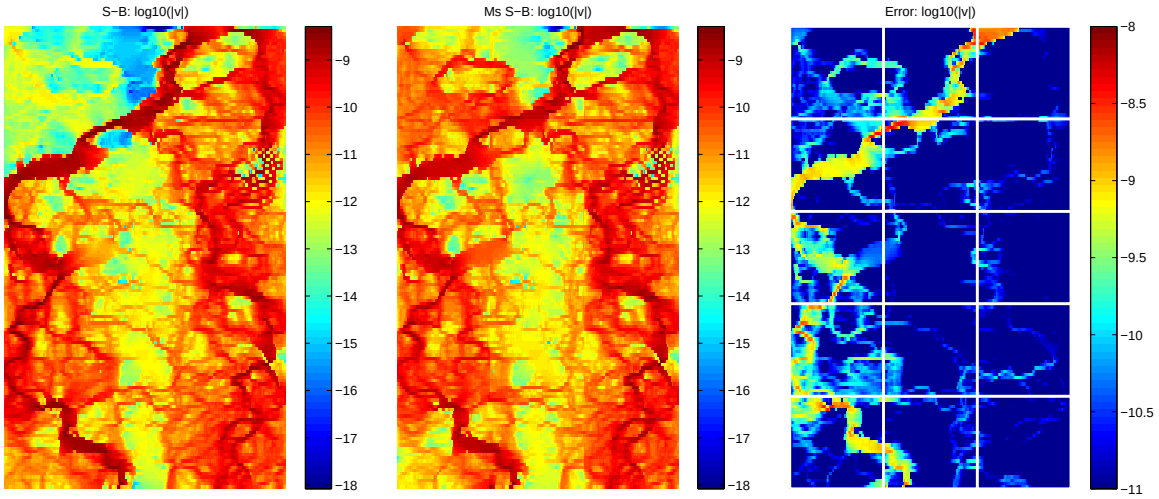

Figure 3 *Velocity magnitude computed using the fine scale Stokes–Brinkman equations (left column), multiscale Stokes–Brinkman (middle column) and velocity discrepancy overlaid by coarse grid (right column).*

et al., 2010) needed several improvements to get good results. Multiscale methods for non-Darcy equations are far from plentiful, so to promote development we include a multiscale example and explain the main challenges with suggested improvements.

Case 1 (2D cut) *The simulation is done on the 2D cut shown in Figure 1. In this 2D cut, the vugs are not connecting the upper and lower edges of the model, as opposed to the full 3D model. We set* $k_f = \infty$ *in the vugs to model them as free-flow regions, and apply a pressure drop from the top to the bottom of the model. Then, we compute the effective permeability using both fine-scale Stokes-Brinkman and multiscale Stokes-Brinkman with a coarse grid of* 3×5 *blocks. The resulting effective permeabilities were* $1.29 \cdot 10^{-2}$ *mD in the fine-scale case and* $1.0 \cdot 10^{-2}$ *mD in the multiscale case.*

Consequently, we see that the multiscale method underestimates the flow through the sample compared with the fine-scale result; this is also seen in the plots of the velocity magnitudes in Figure 3. However, as already mentioned, the multiscale solver is generally good at predicting flow paths, even though it underestimates flow density and in some cases overestimates the flow through low-permeable areas, as seen in the upper-left corner of the velocity plot. Figure 3 shows that the main velocity error not surprisingly is concentrated in the high-flow regions, and in particular where the coarse blocks split main flow paths or where the connection between the vugs is narrow.

These results are typical for numerous simulations run with different coarse grid resolutions. We recognize that there are three main areas where the method needs further development. First of all, the coarse grid does not adapt to the high-flow regions, in this case the vugs. This is a well-known potential source of errors in multiscale simulations and can be solved by adapting the coarse grid to the high-flow regions (Aarnes et al., 2006), or by extending the region where the basis functions are valid, using so-called overlap. One other solution could be to introduce additional basis functions for long range structures, as proposed by Natvig et al. (2009). Unfortunately, our simple prototype implementation is incapable of handling the irregular shaped vugs found in the 2D cuts.

Second, we also notice that the error is large where the vugs are on the pressure boundary. This problem was also reported in (Gulbransen et al., 2010) and is related to the way the basis functions are computed; the local flow problems are set up by assigning sources scaled by the permeability.

Lastly, one of the main principles of the MsMFE method is that the basis functions reproduce unit flow over the coarse grid boundaries. For the Darcy equations, this is assured by using the lowest-order Raviart–Thomas element. When we apply the Taylor–Hood element for the Stokes–Brinkman mulitscale basis functions, we are not able to exactly reproduce a unit flow over the coarse edges. Thus, we introduce a small error in the velocity field near the coarse-block boundaries. This leads to an underestimation of the flow rate that will grow with increasing flow. A solution to this problem is to use a pressure element with nodes in the cell centers.

From the above discussion, we conclude that the multiscale Stokes–Brinkman method needs further improvements before it can be applied for full 3D simulations of vuggy media. However, before continuing the development of the method, we did an analysis to quantify the importance of using the Stokes–Brinkman equations for the particular problem of computing the effective permeability of a 3D model. The results from this study, presented in the next section, did not encourage us to put more effort into the multiscale Stokes–Brinkman method.

Parameter Study

In this section, we will analyze the importance of using the Stokes–Brinkman equations, as opposed to only the Stokes (or Darcy) equations, to compute an accurate effective permeability of the 3D rock sample. Our analysis will rely entirely on the assumption that all we need to capture on the coarse scale is the effective permeability.

We have performed a computational parameter study on several idealized rectangular models in 2D (Krotkiewski et al., 2010). Herein, we only report results for the setup that was most relevant to our particular 3D model: a square domain of width L with a through-going (percolating) crack of width ℓ that touches both boundaries. The fluid flow is driven by a horizontal pressure difference Δp. The boundary conditions at $y = 0$ and $y = 1$ are set to no-slip (v_x=0, v_y=0) to enable us to study the transition between the Darcy and Stokes regimes (see (Krotkiewski et al., 2010) for an in-depth explanation of the boundary conditions). In our model, we assume that the crack is a free-flow region with permeability $k_f = \infty$. We then systematically vary the matrix permeability and solve the flow problem using the Stokes–Brinkman equations (1). The effective permeability can then be computed from Darcy's law over the REV

$$k_{\text{eff}} = -\frac{q\mu L}{\Delta p}, \quad \text{where} \quad q = \frac{1}{L}\int_0^L v_x(L, x)\, dy. \tag{2}$$

For large matrix permeabilities (i.e., both the fracture, and the matrix are free-flow domains), this setup corresponds to 2D Poiseuille flow in a channel, for which the effective permeability is given by

$$k_S = \frac{L^2}{12}. \tag{3}$$

Case 2 (Through-Going Fracture) *The effect of a long-range channel/fracture is modeled as a rectangle of width ℓ connecting the left and right boundaries of a box of width L. The fracture is modeled as a free-flow, large aspect-ratio rectangle surrounded by the matrix. Because $k_f = \infty$ in the fracture, the Darcy term in the Stokes–Brinkman equations is zero and we only solve the Stokes equations here. In the matrix, the full Stokes–Brinkman equations are solved. In the case of a low-permeable matrix, the effective permeability of the entire domain is expected to be governed by the permeability of the free-flow (Stokes) domain.*

Using the Stokes–Brinkman equations and varying the matrix permeability k_m , we compute the effective permeability k_{eff} in the transition zone between the end-member cases to analyze the influence of the matrix permeability on the regime of the flow. Figure 4 shows k_{eff} normalized by k_S for the two cases of a through-going fracture compared to the results for a confined high-permeability fracture (Krotkiewski

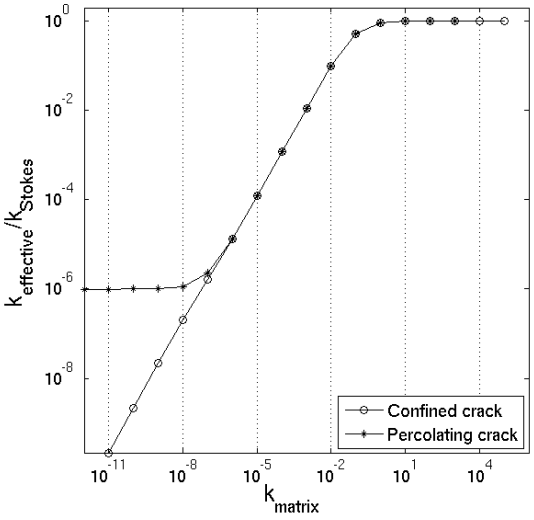

Figure 4 *Comparison of the effective permeability of a through-going and a confined fracture in a square domain with a fracture of diameter* $\ell = 0.01$.

et al., 2010). For the through-going fracture with $k_m \ll k_f$*, the effective permeability of the domain is constant and independent of* k_m*; that is, the velocities in the entire low-permeable matrix domain are negligible compared to the velocities in the fracture. In this case, the effective permeability given by* (3) *is modified by the volume fraction of the fracture. In the studied scenario, this means that*

$$\frac{k_{\mathit{eff}}}{k_S} = \frac{(\ell^2/12)(\ell/L)}{L^2/12} = \frac{0.01^3/12L}{L^2/12} = 10^{-6},$$

which is in agreement with the results from Figure 4. Moreover, it is clear from Figure 4 that for a through-going fracture with $k_f \approx 10^{-5}$ *and* $k_m \leq 10^{-9}$*, the porous matrix does not contribute much, and the effective permeability of the entire domain can be obtained by a Stokes flow simulation. This insight can be used to compute the effective permeability of the 3D model presented earlier.*

Computing Effective Permeabilities

While in all the 2D cuts the vugs are confined in the matrix, Figure 2 shows that they do connect to the opposite sides of the domain in 3D, which illustrates the importance of 3D simulations. We can therefore use the insights obtained from the analysis of the through-going fracture discussed in the parameter study in the previous section to simplify our computational setup. The relatively impermeable matrix is not expected to significantly affect the effective permeability of this rock sample. Therefore, to compute k_{eff} we have removed the porous matrix from the model. The fact that we are able to do so is crucial from the point of view of the iterative solver. Large permeability ratios in the matrix, and the Stokes regime on a different spatial scale lead to an ill-condition system that is virtually impossible to solve. On the other hand, by considering only the Stokes domain, we are dealing with channel flow of a homogeneous fluid—a task that is much simpler for an iterative method.

Case 3 (Full 3D Model) *In the simulations, we apply ambient pressure gradient in the vertical direction. For the nodes lying on the channel boundary (marked in yellowish color in Figure 2), we set no-slip boundary condition. The mesh consists of around 800 000 cells (eight million nodes), and it took three hours to solve the system on 360 Opteron cores of a Cray XT4 cluster. Figure 5 shows the flow channels and fluid velocity through the vugs. For the purpose of visualization, we have placed half a million passive markers at* $z = 0$*. The markers have been advected using the computed velocity field and a timestep chosen so that they are not transported beyond their neighbor cells. The markers are colored red in the high-velocity regions and blue in the low-velocity regions. Some void spaces do not contain any*

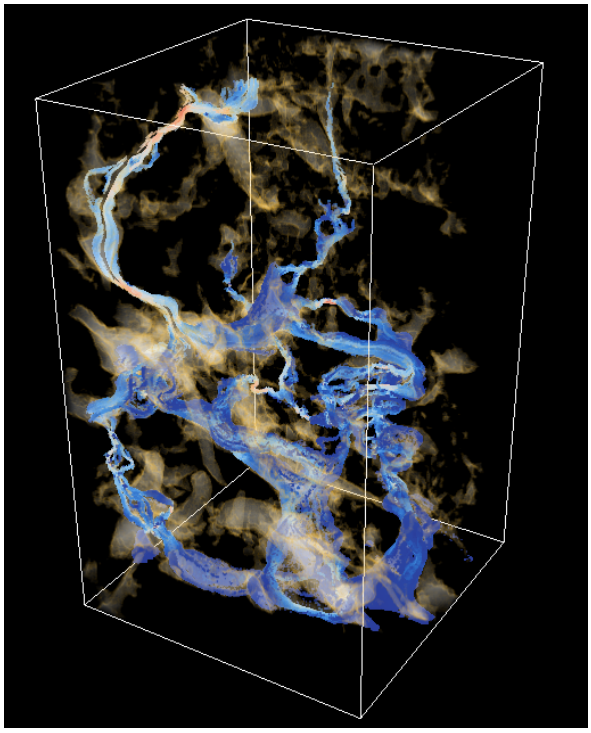

Figure 5 *Stokes flow inside the fractures and vugs, without considering the porous matrix. Blue color of the streamlines denotes low velocity, red color marks high velocity. The model has 800 000 cells and eight million nodes.*

markers—this indicates the stagnant zones. Clearly, studying the actual flow pattern is important if one wants to analyze e.g., the residual saturation. The effective permeability obtained from this calculation is $7.9 \cdot 10^5$ *mD, i.e., much higher than the permeability of the surrounding porous matrix. This result further indicates that it is not necessary to take the Darcy matrix into account in this case. As mentioned previously, the highest permeability in the porous matrix is roughly* 10^3 *mD, while the average is much lower.*

Concluding Remarks

In this paper, we have studied the effective permeability of a medium composed of long-range fractures and channels surrounded by a porous matrix with lower permeability. Through analysis of a synthetic 2D case we found that effective permeability in a domain with a large, through-going free-flow fracture surrounded by a low-permeable matrix could be computed using a Stokes flow solver, and hence there was no need to use the full Stokes–Brinkman model. Using a pure Stokes solver is advantageous from a numerical perspective, since the resulting system of equations is better conditioned. When the permeability of the fracture $k_f \geq 10^4 k_m$, the velocities in the Darcy matrix are negligible and do not significantly change neither the effective permeability nor the flow pattern. Using this observation, we were able to compute the effective permeability of a high-resolution rock model coming from a volumetric CT-scan with more than eight million voxels.

Acknowledgments

This study has mostly been performed using computational facilities provided by the Norwegian Metacenter for Computational Science (NOTUR). The work of Krotkiewski and Schmid was supported by a Center of Excellence grant from the Norwegian Research Council to PGP (Physics of Geological Processes) at the University of Oslo. The authors would like to thank Marcin Dabrowski for insightful discussions. The research of Ligaarden and Lie was funded in part by Shell Norge AS and the Research Council of Norway through grants no. 175962 and 186935.

Lie also acknowledges partial funding from the Center of Mathematics for Applications, University of Oslo. The Pipe Creek CT-scan data was originally collected by the Bureau of Economic Geology at The University of Texas at Austin with funding from the Industrial Associates of the Reservoir Characterization Research Laboratory. The authors are grateful to Bob Loucks, Chris Zahm, and Jim Jennings for assistance accessing the data.

References

Aarnes, J.E., Krogstad, S. and Lie, K.A. [2006] A hierarchical multiscale method for two-phase flow based upon mixed finite elements and nonuniform coarse grids. *Multiscale Model. Simul.*, **5**(2), 337–363 (electronic), ISSN 1540-3459.

Arbogast, T. and Brunson, D.S. [2007] A computational method for approximating a Darcy–Stokes system governing a vuggy porous medium. *Comput. Geosci.*, **11**(3), 207–218.

Arbogast, T. and Bryant, S.L. [2002] A two-scale numerical subgrid technique for waterflood simulations. *SPE J.*, **7**(4), 446–457.

Arbogast, T. and Gomez, M.S.M. [2009] A discretization and multigrid solver for a Darcy–Stokes system of three dimensional vuggy porous media. *Comput. Geosci.*, **13**(3), 331–348, doi: 10.1007/s10596-008-9121-y.

Arbogast, T. and Lehr, H.L. [2006] Homogenization of a Darcy–Stokes system modeling vuggy porous media. *Comput. Geosci.*, **10**(3), 291–302.

Beavers, G.S. and Joseph, D.D. [1967] Boundary conditions at a naturally permeable wall. *Journal of Fluid Mechanics*, **30**, 197–207, doi:10.1017/S0022112067001375.

Brinkman, H.C. [1947] A calculation of the viscous force exerted by a flowing fluid on a dense swarm of particles. *Applied Scientific Research Section a-Mechanics Heat Chemical Engineering Mathematical Methods*, **1**(1), 27–34.

Chen, Z. and Hou, T. [2003] A mixed multiscale finite element method for elliptic problems with oscillating coefficients. *Math. Comp.*, **72**, 541–576.

Dabrowski, M., Krotkiewski, M. and Schmid, D.W. [2008] MILAMIN: MATLAB-based finite element method solver for large problems. *Geochem. Geophys. Geosyst.*, **9**(1), doi:doi:10.1029/2007GC001719.

Elman, H., Silvester, D. and Wathen, A. [2005] *Finite Elements and Fast Iterative Solvers (with Applications in Incompressible Fluid Dynamics)*. Numerical Mathematics and Scientific Computation, Oxford University Press.

Gulbransen, A.F., Hauge, V.L. and Lie, K.A. [2010] A multiscale mixed finite element method for vuggy and naturally fractured reservoirs. *SPE J.*, **15**(2), doi:10.2118/119104-PA.

Hou, T. and Wu, X.H. [1997] A multiscale finite element method for elliptic problems in composite materials and porous media. *J. Comput. Phys.*, **134**, 169–189.

Jennings, J.W. and Lucia, F.J. [2003] Predicting permeability from well logs in carbonates with a link to geology for interwell permeability mapping. *SPE Reservoir Evaluation & Engineering*, **6**(4), 215–225.

Jenny, P., Lee, S.H. and Tchelepi, H.A. [2003] Multi-scale finite-volume method for elliptic problems in subsurface flow simulation. *J. Comput. Phys.*, **187**, 47–67.

Karper, T., Mardal, K.A. and Winther, R. [2009] Unified finite element discretizations of coupled Darcy–Stokes flow. *Numer. Meth. PDEs*, **25**(2), 311–326, doi: 10.1002/num.20349.

Kippe, V., Aarnes, J.E. and Lie, K.A. [2008] A comparison of multiscale methods for elliptic problems in porous media flow. *Comput. Geosci.*, **12**(3), 377–398, ISSN 1420-0597, doi:10.1007/s10596-007-9074-6.

Krotkiewski, M., Ligaarden, I., Schmid, D.W. and Lie, K.A. [2010] On the importance of the Stokes–Brinkman equations for computing effective permeability in carbonate karst reservoirs. Submitted.

Liu, S. and Masliyah, J.H. [2005] Dispersion in porous media. In: Vafai, K. (Ed.) *Handbook of porous media*. CRC Press, 2nd edn., 110.

Natvig, J.R. et al. [2009] Multiscale mimetic solvers for efficient streamline simulation of fractured reservoirs. *SPE Reservoir Simulation Symposium, The Woodlands, TX, USA, 2–4 February 2009*, doi:10.2118/119132-MS.

Popov, P., Qin, G., Bi, L., Efendiev, Y., Kang, Z. and Li, J. [2009a] Multiphysics and multiscale methods for modeling fluid flow through naturally fractured carbonate karst reservoirs. *SPE Reservoir Evaluation & Engineering*, **12**(2), 218–231.

Popov, P., Efendiev, Y. and Qin, G. [2009b] Multiscale modeling and simulations of flows in naturally fractured karst reservoirs. *Commun. Comput. Phys.*, **6**, 162–184.

Renard, P. and de Marsily, G. [1997] Calculating equivalent permeability: A review. *Advances in Water Resources*, **20**(5–6), 253–278.

Willems, J. [2009] *Numerical Upscaling for Multiscale Flow Problems*. Ph.D. thesis, Technische Universität Kaiserslautern.

Zhang, L., Bryant, S.L., Jennings, J.W., Arbogast, T.J. and Paruchuri, R. [2004] Multiscale flow and transport in highly heterogeneous carbonates. *Proceedings of the SPE Annual Technical Conference and Exhibition*, sPE 90336.